

MULTIPLICITY FORMULA AND STABLE TRACE FORMULA

ZHIFENG PENG

ABSTRACT. Let G be a connected reductive group over \mathbb{Q} . In this paper, we will stabilize the local trace formula, in particular, we construct the explicit form of the spectral side of stable local trace formula in the Archimedean case, when one component of the test function is cuspidal. Then we will also give the multiplicity formula for discrete series. At the same time, we obtain the stable version of L^2 -Lefschetz number formula.

1. INTRODUCTION

Suppose that G is a connected reductive group over \mathbb{Q} , and that Γ is an arithmetic subgroup of $G(\mathbb{R})$ defined by congruence conditions. Consider the regular representation R with $G(\mathbb{R})$ acting on $L^2(\Gamma \backslash G(\mathbb{R}))$ through right translation. The fundamental problem is to decompose R into a direct sum of irreducible representations. In general, we decompose R into two parts

$$R = R_{\text{disc}} \oplus R_{\text{cont}},$$

where R_{disc} is the sum of discrete series, and R_{cont} is the continuous spectrum. The continuous spectrum can be understood by Eisenstein Series, which was studied by Langlands [22]. We only need to study R_{disc} . If $\pi \in R_{\text{disc}}$ is an irreducible representation, we denote $R_{\text{disc}}(\pi)$ for the π -isotypical subspace of R_{disc} . Then

$$R_{\text{disc}}(\pi) = \pi^{\oplus m_{\text{disc}}(\pi)},$$

where $m_{\text{disc}}(\pi)$ is the multiplicity. The classical problem is to find a finite closed formula for $m_{\text{disc}}(\pi)$.

If π belongs to the square integrable discrete series, and $\Gamma \backslash G$ is compact, then Langlands [21] gave a formula for $m_{\text{disc}}(\pi_{\mathbb{R}})$. If $\Gamma \backslash G$ is noncompact, the first result is for $G = SL_2(\mathbb{R})$, the formula of $m_{\text{disc}}(\pi_{\mathbb{R}})$ appeared in Selberg's paper [26]. For G having \mathbb{R} -rank one, there is a formula for it in [25]. Generally, for G of any \mathbb{R} -rank, Arthur [23] has studied the sum of multiplicities

$$(1.1) \quad \sum_{\pi \in \Pi_{\text{disc}}(\mu)} m_{\text{disc}}(\pi_{\mathbb{R}}, K_0)$$

by using the invariant trace formula, where the L -packet $\Pi_{\text{disc}}(\mu)$ is a finite set of discrete series representations with the same infinitesimal character μ , which is an irreducible finite dimensional representation of G . More generally, we can consider Hecke operators h on

$L^2(\Gamma \backslash G)$ that commute with R , and write $R_{\text{disc}}(\pi, h)$ for the restriction of h to $R_{\text{disc}}(\pi)$. Arthur [3] found a formula for

$$(1.2) \quad \sum_{\Pi_{\text{disc}}(\mu)} \text{tr}(R_{\text{disc}}(\pi, h)),$$

under a weak regularity condition. The spectral side of invariant trace formula corresponds to (1.2), if the test function is a stable cuspidal function f_μ . Therefore, the explicit formula for (1.2) follows from the geometric side of invariant trace formula. A key point is that the invariant distribution $I_M(\gamma, f_\mu)$ vanishes, if γ is not semisimple.

We shall give a formula for the multiplicity of single representation $m_{\text{disc}}(\pi)$ and $\text{tr}(R_{\text{disc}}(\pi, h))$, which was conjectured by Spallone and Wakatsuki [31, Conjecture 1] who also had checked two special cases. If we use the invariant trace formula to obtain the formula, with the test function being the pseudo-coefficient f_{π_μ} , which is a cuspidal function, but not a stable function. So the invariant distribution $I_M(\gamma, f_{\pi_\mu})$ in general does not vanish for γ with non-trivial unipotent part. To capture the single representation and obtain the stable cuspidal function, we need to use the endoscopy theory and stable trace formula.

Fortunately, when G is K -group, Arthur [9], [10], [11] had obtained the stabilization of the general trace formula in 2003, assuming the Fundamental Lemma. In 2008, Ngo [20] solved this difficult problem. So we got the unconditional stabilization of global and local trace formula. From this we obtain the following proposition.

Proposition 1.1. *For any $h \in \mathcal{H}(G(\mathbb{A}_{\text{fin}}))$, $f_{\pi_\mu} \in C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, we have*

$$(1.3) \quad \begin{aligned} \text{tr}(R_{\text{disc}}(\pi_\mu, h)) &= I(f_{\pi_\mu} h) \\ &= \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \sum_{M' \in \mathcal{L}^{G'}} |W_0^{M'}| |W_0^{G'}|^{-1} \sum_{\delta \in \Delta(M', V, \zeta)} b^{M'}(\delta) S_{M'}^{G'}(\delta, f_{\pi_\mu}) (h_M)^{M'}(\delta) \end{aligned}$$

See section 4 for more details. However, the stable coefficient $b^{M'}(\delta)$ and the stable distribution $S_{M'}^{G'}(\delta, f_{\pi_\mu})$ are not explicit. Using the fact that the transfer function $f_{\pi_\mu}^{G'}$ is stable cuspidal function for endoscopic group G' of G , we obtain the vanishing of $S_{M'}^{G'}(\delta, f_{\pi_\mu})$ for δ not semisimple. It remains to study the semisimple elements of $G'(\mathbb{R})$, which are the main study object of the local trace formula. We shall give the explicit formula for the stable distribution $S_{M'}^{G'}(\delta, f_{\pi_\mu})$ through comparing the stable local trace formula with stable Weyl integral formula. We need to directly stabilize the spectral side of the local trace formula in the Archimedean place.

In this case, where the test function is cuspidal on the two places in the p -adic case, Arthur [7] gave a concrete formula about the geometric side of stable local trace formula, which is just an inner product. We shall establish a formula in the Archimedean case, with the test function being cuspidal at only one place, so it is more complicated. Generally, to stabilize the trace formula, the key point is that the stable distribution of endoscopy group G' is independent of G . So we only need to study the stable distribution on quasisplit groups. Our work relies on harmonic analysis on reductive groups of Harish-Chandra

[13], [14], [15], and Langlands [22] classification of the irreducible representations of real algebraic groups. We also need the work of Shelstad [29], [30], who classified the tempered representations, directly constructed the spectral transfer factors, and gave the inverse adjoint relations. We will directly build the transfer factors and obtain the explicit formula of the spectral side of the local trace formula.

Theorem 1.2. *If $f = f_1 \times \bar{f}_2$, $f_1 \in C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, $f_2 \in C(G(\mathbb{R}), \zeta)$, then*

$$\begin{aligned} I_{\text{disc}}(f) &= \int_{T_{\text{ell}}(G, \zeta)} i^G(\tau) f_{1,G}(\tau) \overline{f_{2,G}(\tau)} d\tau \\ &= \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \widehat{S}^{G'}(f'), \end{aligned}$$

and $\widehat{S}^{G'}(f')$ is stable distribution, where

$$\begin{aligned} i^G(\tau) &= |d(\tau)|^{-1} |R_{\pi, r}|^{-1}, \\ \widehat{S}^{G'}(f') &= \int_{\Phi_2(G', \zeta)} S^{G'}(\phi') \widetilde{f}'_1(\phi') \overline{f'_2(\phi')} d\phi', \\ S^{G'}(\phi') &= \frac{1}{|\mathcal{S}_{\phi'}|}, \quad \phi = \xi' \circ \phi'. \end{aligned}$$

We shall obtain the local stable trace formula $S_{\text{geo}} = S_{\text{spec}}$ Theorem in 7.1 and an explicit formula for stable distribution. We then have the following main theorem.

Theorem 1.3. *If $h \in \mathcal{H}(G(\mathbb{A}_{\text{fin}}))$, and the highest weight of representation μ is regular, then we have*

$$\begin{aligned} &\text{tr}(R_{\text{disc}}(\pi_{\mathbb{R}}, h)) \\ &= \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \sum_{M' \in \mathcal{L}^{G'}} (-1)^{\dim(A_{M'}/A_{G'})} |W_0^{M'}| |W_0^{G'}|^{-1} \sum_{\delta \in \{M'(\mathbb{Q})\}} P_{\mu}(M') S\Phi_{M'}(\phi'_{\mu}, \delta) (h_M)^{M'}(\delta), \end{aligned}$$

and the multiplicity formula of the discrete series is

$$m_{\text{disc}}(\pi_{\mathbb{R}}, K_0) = \text{tr}(R_{\text{disc}}(\pi_{\mathbb{R}}, 1_{K_0})).$$

We will also give the stable L^2 -Lefschetz formula, when the test function is stable cuspidal. In this case, In Kottwitz's preprint [19] shall give another treatment to directly stabilize Arthur's trace formula [3].

Theorem 1.4. *For any $h \in \mathcal{H}(G(\mathbb{A}_{\text{fin}}))$, we have*

$$\mathcal{L}_{\mu}(h) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \sum_{M' \in \mathcal{L}^{G'}} |W_0^{M'}| |W_0^{G'}|^{-1} \sum_{\delta \in \{M'(\mathbb{Q})\}} F_{\mu}(M') S\Phi_{M'}(\phi', \delta) (h_M)^{M'}(\delta).$$

The content of paper is as follows, In section 2, I will introduce the K -groups, which are unions of the reductive group. If F is a p -adic field, then the K -group is still connected reductive group. However, if F is an Archimedean field, then the K -group is not connected. But the K -group is a good object for stabilizing the trace formula, and any connected

reductive group G_1 is a component of a unique K -group G . We will give an example for the K -group, for which the invariant and stable distribution can be extended to K -group as in [8].

In section 3, we will obtain the relation between multiplicity and invariant trace formula. But when the test function is a pseudo-coefficient, we cannot give a concrete invariant trace formula, since the test function is not stable, we can not cancel the contribution of unipotent element of the invariant of the distribution. So we need to stabilize the invariant trace formula to overcome this obstacle. In general, the invariant trace formula is the identity obtained from two different expansions of a certain linear form $I(f)$. One expansion is the geometric expansion

$$I(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in \Gamma(M, S)} a^M(S, \gamma) I_M(\gamma, f),$$

which is a linear combination of distributions parameterized by conjugacy classes γ in Levi subgroups $M(F_S)$. The other expansion is the spectral expansion

$$I(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Pi(M, S)} a^M(S, \pi) I_M(\pi, f) d\pi,$$

which is a linear combination of distributions parameterized by representations π of Levi subgroup $M(F_S)$. Where $f \in \mathcal{H}(G, S)$ in the Hecke algebra of $G(F_S)$ (see [1], [2]), S is the finite set of place in F . Arthur [11] has stabilized the invariant trace formula. So we can expand the multiplicity by using the geometric side of global of stable trace formula. In section 4, We can use the splitting formula to reduce the local component of global trace formula to Archimedean case. Compared to the local trace formula, the local component of global trace formula is more complicated, because it contains the contributions of distributions of nontrivial unipotent elements. But if the test function is stable cuspidal, the nontrivial unipotent distributions vanish, and then the local component of local trace formula coincide with the local component of global trace formula. Moreover, the pseudo-coefficients of representation can be transferred to stable cuspidal functions under the Shelstad transfer mapping. So it is enough for us to study the stabilization of local trace formula. In general, the geometric side of local trace formula concerns the semisimple regular elements, and the spectral side of the local trace formula concerns the tempered representations.

The spectral side of the invariant local trace formula contains a natural object, which is the virtual character. A simple invariant local trace formula can be given by the virtual character in [5]. Therefore, in section 5, we introduce the virtual character in the Archimedean case, and define the transfer factors $\Delta(\tau, \phi)$ and $\Delta(\phi, \tau)$, which includes Shelstad's work [29]. Then we stabilize the spectral side of the local trace formula. When one component of the test function is cuspidal, we just need to consider the elliptic representations. In section 6, we obtain a formula for the spectral side of stable of local trace formula.

In section 7, we study the main term $S_M^G(\delta, \phi)$. To do this, we need to stabilize the Weyl integral formula, which connects the geometric and spectral sides of the local component

of local trace formula. We can then compare the stable of local trace formula, and the main term of the formula will appear.

In section 8, we establish the relation between $S_M^G(\delta, f)$ and the invariant main term $\Phi_M(\gamma, f)$, then we can overcome the key obstacle. We have $S_M^G(\delta, f_{\phi_\mu}) = 0$, if δ is not semisimple, f_{ϕ_μ} is a stable cuspidal function which is transferred by a pseudo-coefficient of π_μ . We shall collect various terms, and they will be combined into our main formula in Theorem 8.3. In section 9, we shall give a stable formula for L^2 -Lefschetz number, when the test function is stable cuspidal.

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2. PRELIMINARIES AND DISTRIBUTIONS OF K-GROUPS

2.1. Notation. Let G be a reductive group over \mathbb{Q} , M be a Levi subgroup of G , P be a parabolic subgroup, and A_G be the \mathbb{Q} -split component of the center of G . If γ is a semisimple element of G , we denote by $G(\mathbb{Q}, \gamma)$ the centralizer of γ in $G(\mathbb{Q})$, and write G_γ for the identity component of $G(\mathbb{Q}, \gamma)$. Write G_{der} for the derived group of G , G_{sc} for the simply connected cover of G_{der} . We say G is cuspidal, if $G(\mathbb{R})$ contains a maximal \mathbb{R} -torus T such that T/A_G is anisotropic over \mathbb{R} , in other words, the \mathbb{Q} -split component coincide with \mathbb{R} split component $A_{G(\mathbb{R})}$ and the real group $G(\mathbb{R})$ contains a \mathbb{R} -elliptic maximal torus. A torus T in G is elliptic if T/A_G is anisotropic, an element of $G(\mathbb{R})$ is elliptic if it is contained in an elliptic torus of G . Let $X^*(G) = \text{Hom}(G, \mathbb{R})$ and $a_G = \text{Hom}(X^*(G), \mathbb{R})$ for the Lie algebra of $A_G(\mathbb{R})$.

We denote by \mathbb{A} the ring of Adèles of \mathbb{Q} , and denote by \mathbb{A}_{fin} the finite part of Adèles ring over \mathbb{Q} , so that $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\text{fin}}$. We denote by G' the endoscopy group of G , denote by $Z(G)$ the center of G . Z stands for a central induced torus in G which means an induce torus over \mathbb{Q} , together with embeddings $Z \hookrightarrow Z(G)$. Throughout the paper, F will be a field of characteristic 0.

2.2. K-groups. K -groups are natural objects for studying the stabilization of general trace formula, which contain several connected components. To work with several groups simultaneously is suitable for studying the transfer properties of the various objects in the trace formula. The use of several inner forms is due originally to Vogan. When Kottwitz learned of Vogan's idea, he applied it to the Langlands-Shelstad transfer factors. We shall follow Arthur's discussion [8] to recall the basic facts about the K -groups, where he extended the geometric transfer factors based on constructions due to Kottwitz.

Definition 2.1. G is called a K -group over a local field F , if

- (1) G is an algebraic variety whose connected components are reductive algebraic groups over F , endowed with an equivalence class of objects $\{(\psi, u)\}$. Here $(\psi, u) = \{(\psi_{\alpha\beta}, u_{\alpha\beta}) : \alpha, \beta \in \pi_0(G)\}$, $\psi_{\alpha\beta} : G_\beta \rightarrow G_\alpha$ is an isomorphism over \bar{F} , and $u_{\alpha\beta} : \Gamma \rightarrow G_{\alpha, \text{sc}}$ is a 1-cocycle, which maps from the Galois group $\Gamma = \text{Gal}(\bar{F}/F)$ to $G_{\alpha, \text{sc}}$. We require that $\{(\psi, u)\}$ satisfy the compatibility conditions,

- (i) $\psi_{\alpha\beta}\tau(\psi_{\alpha\beta})^{-1} = \text{Int}(u_{\alpha\beta}(\tau));$
 - (ii) $\psi_{\alpha\gamma} = \psi_{\alpha\beta}\psi_{\beta\gamma};$
 - (iii) $u_{\alpha\gamma}(\tau) = \psi_{\alpha\beta,sc}(u_{\beta\gamma}(\tau))u_{\alpha\beta}(\tau),$ for any $\alpha, \beta, \gamma \in \pi_0(G)$ and $\tau \in \Gamma,$
- (2) the corresponding sequences

$$\{1\} \longrightarrow \{u_{\alpha\beta} : \beta \in \pi_0(G)\} \longrightarrow H^1(F, G_\alpha) \xrightarrow{K_{G_\alpha}} \pi_0(Z(\widehat{G})^\Gamma)^*.$$

of pointed sets are exact. Here $\alpha \in \pi_0(G)$, the map K_{G_α} is defined in [17, §1].

The notation $\pi_0(G)$ is a set of indices for the components of G , but we write $\pi_0(Z(\widehat{G})^\Gamma)$ for the set of connected components of $Z(\widehat{G})^\Gamma$ as usual.

We say that two such families (ψ, u) and (ψ', u') are equivalent, if there are elements $g_{\alpha\beta} \in G_{\alpha,sc}$ such that $\psi'_{\alpha\beta} = \text{Int}(g_{\alpha\beta})\psi_{\alpha\beta}$ and $u'_{\alpha\beta}(\tau) = g_{\alpha\beta}u_{\alpha\beta}(\tau)\tau(g_{\alpha\beta})^{-1}$, for any $\alpha, \beta \in \pi_0(G)$ and $\tau \in \Gamma$. We call a representative (ψ, u) from the equivalence class as a frame for G . If F is p -adic, K_{G_α} is a bijection [17, Theorem 1.2], a K -group is therefore just a connected reductive group. If F is Archimedean, the kernel of K_{G_α} is the image of $H^1(F, G_{\alpha,sc})$ in $H^1(F, G_\alpha)$ [17, Theorem 1.2], and the number of components of a K -group over \mathbb{R} therefore is equal to the number of classes in this image.

Suppose that G is a K -group, then we can write $G = \coprod_{\alpha \in \pi_0(G)} G_\alpha$. A homomorphism between K -groups G and \bar{G} over F is a morphism

$$\theta = \coprod_{\alpha} (\theta_\alpha : G_\alpha \rightarrow \bar{G}_{\bar{\alpha}})$$

from G to \bar{G} (as varieties over F) that preserves all the structure. In other words, it satisfies the following two properties,

- (1) For any $\alpha \in \pi_0(G)$, and $\bar{\alpha} = \theta(\alpha)$ the image of α in $\pi_0(\bar{G})$, the restriction $\theta_\alpha : G_\alpha \rightarrow \bar{G}_{\bar{\alpha}}$ is a homomorphism of connected algebraic groups.
- (2) There are frames (ψ, u) and $(\bar{\psi}, \bar{u})$ for G and \bar{G} , such that $\theta_\alpha \circ \psi_{\alpha\beta} = \bar{\psi}_{\bar{\alpha}\bar{\beta}} \circ \theta_\beta$, and $\bar{u}_{\bar{\alpha}\bar{\beta}} = \theta_{\alpha,sc}(u_{\alpha\beta})$, for each $\alpha, \beta \in \pi_0(G)$.

An isomorphism of K -groups is an invertible homomorphism. Arthur introduces the notion of weak isomorphism in [9, §4]. It satisfies all the requirements of an isomorphism except for the condition relating $u_{\bar{\alpha}\bar{\beta}}$ with $u_{\alpha\beta}$, so that one can identify K -groups that differ only by the choice of functions $\{u_{\alpha\beta}\}$. So if we are given a connected reductive group G_1 over F , we can find a K -group over F , such that $G_{\alpha_1} = G_1$ for some $\alpha_1 \in \pi_0(G)$. There could be several such G , but the weak isomorphism class of G is uniquely determined by G_1 . In particular, any connected quasisplit group G^* has a quasisplit inner K -form G , which is unique up to weak isomorphism. We say that K -group G is quasisplit if it has a connected component that is quasisplit over F .

The Levi subgroup M of K -group G was defined [8, §1]. For any such M , we construct the associated objects $W(M)$, $\mathcal{P}(M)$, $\mathcal{L}(M)$ and $\mathcal{F}(M)$ as in [8, §1], which represent Weyl group, the set of the parabolic subgroups which the component of Levi subgroup equals M , the set of Levi subgroups which contain the Levi subgroup M and the set of parabolic subgroups which the component of the Levi subgroups contain the Levi subgroup

M respectively. They play the same role as in the connected case. We can also form a dual group \widehat{G} for G , and a dual Levi subgroup $\widehat{M} \subset \widehat{G}$ for M , we mean a Γ -stable Levi component of a Γ -stable parabolic subgroup of \widehat{G} . For any such group \widehat{M} , we also have the analogue object $\mathcal{P}(\widehat{M})$, $\mathcal{L}(\widehat{M})$ and $\mathcal{F}(\widehat{M})$, with the understanding that the sets contain only Γ -stable elements. It comes with a bijection $L \rightarrow \widehat{L}$ from $\mathcal{L}(M)$ to $\mathcal{L}(\widehat{M})$, and a bijection $P \rightarrow \widehat{P}$ from $P(M)$ to $P(\widehat{M})$.

Invariant harmonic analysis for connected real groups extends in a natural way to K -groups. For example, we have the Harish-Chandra's Schwartz space. For simplicity, we also denote it by Schwartz space

$$C(G) = \bigoplus_{\alpha \in \pi_0(G)} C(G_\alpha)$$

on $G(\mathbb{R})$, and its invariant analogue

$$I(G) = \bigoplus_{\alpha \in \pi_0(G)} I(G_\alpha).$$

Elements in $C(G)$ are the functions on $G(\mathbb{R})$, and elements in $I(G)$ can be regarded as the functions on the disjoint union

$$\Pi_{\text{temp}}(G) = \coprod_{\alpha \in \pi_0(G)} \Pi_{\text{temp}}(G_\alpha)$$

of sets of irreducible tempered representations on the groups G_α , or as functions on the disjoint union

$$\Gamma_{\text{reg}}(G) = \coprod_{\alpha \in \pi_0(G)} \Gamma_{\text{reg}}(G_\alpha)$$

of the sets of strongly regular conjugacy classes in the groups $G_\alpha(\mathbb{R})$.

For purpose of induction, it is convenient to fix a central character datum (Z, ζ) for G , where Z is an induced torus over \mathbb{R} , with central embedding $Z \rightarrow Z_\alpha \subset G_\alpha$ that are compatible with isomorphisms $\psi_{\alpha\beta}$, the second component ζ is a character on $Z(\mathbb{R})$, which transfers to a character ζ_α on $Z_\alpha(\mathbb{R})$ for each α .

We can then form the space

$$C(G, \zeta) = \bigoplus_{\alpha \in \pi_0(G)} C(G_\alpha, \zeta_\alpha)$$

of ζ^{-1} equivariant Schwartz functions on $G(\mathbb{R})$. And its invariant analogue

$$I(G, \zeta) = \bigoplus_{\alpha \in \pi_0(G)} I(G_\alpha, \zeta_\alpha).$$

Elements in $I(G, \zeta)$ may be regarded either as ζ^{-1} -equivariant functions on $\Pi_{\text{temp}}(G, \zeta)$ or on $\Gamma_{\text{reg}}(G/Z)$. We can obtain the corresponding objects analogously.

If γ lies in $\Gamma(G_\alpha)$, we write G_γ for the centralizer $G_{\alpha, \gamma}$ in G_α of (some representative of) γ . The two classes γ_1 and γ_2 in $\Gamma(G)$ with $\gamma_i \in \Gamma(G_{\alpha_i})$ for $i = 1, 2$ are stable conjugate

if $\psi_{\alpha_1\alpha_2}$ is conjugate in $G_{\alpha_1}(\bar{F})$ to γ_1 , for any frame (ψ, u) . We can then write $\Delta_{\text{reg}}(G(F))$ for the set of strongly regular stable conjugacy classes in $G(F)$. There is a canonical injection $\delta \rightarrow \delta^*$ from $\Delta_{\text{reg}}(G)$ to the set $\Delta_{\text{reg}}(G^*) = \Delta_{\text{reg}}(G^*(F))$ of strongly regular stable conjugacy classes in quasisplit inner twist $G^*(F)$.

An endoscopic datum for G is defined entirely in terms of the dual group \widehat{G} , and is therefore no different from the connected case. $\mathcal{E}(G)$ will stand for the set of isomorphism classes of endoscopic data for G that are relevant to G . An element in $\mathcal{E}(G)$ is therefore the image of some elliptic endoscopic datum $M' = (M', \mathcal{M}', s', \zeta')$ in $\mathcal{E}_{\text{ell}}(M)$, for a Levi subgroup M of G and a dual Levi subgroup \widehat{M} of \widehat{G} . Where elliptic means that the image of \mathcal{M}' in ${}^L M$ is contained in no proper parabolic subgroup ${}^L M$, or equivalently that $(Z(\mathcal{M}')^\Gamma)^0 = (Z(\widehat{M})^\Gamma)^0$. The set $\mathcal{E}(G)$ embeds into the larger set $\mathcal{E}(G^*)$, which we identify with the collection of all isomorphism classes of endoscopic data for G . For each $G' \in \mathcal{E}(G^*)$, we fix a central extension as in [7, §2].

$$1 \longrightarrow \widetilde{Z}' \longrightarrow \widetilde{G}' \longrightarrow G' \longrightarrow 1$$

of G' by a central induced torus of \widetilde{Z}' , then there exists an L -morphism $\widetilde{\zeta}' : \mathcal{G}' \rightarrow {}^L \widetilde{G}'$.

A K -group is a natural domain for the transfer factors of [24]. If F is Archimedean, Arthur extends the transfer factors to the K -groups. It is well known [30] that the set of conjugacy classes in the stable conjugacy classes of K -group can be parametrized by the set $\mathcal{E}(T) = \text{Im}(H^1(\Gamma, T_{\text{sc}}) \rightarrow H^1(\Gamma, T))$, where T is the maximal torus of G . However, if G is a connected group, it's just parametrized by a subset $\mathcal{D}(T)$ of $\mathcal{E}(T)$. Here $\mathcal{D}(T) = \text{Ker}(H^1(\Gamma, T) \rightarrow H^1(\Gamma, G))$, where T is a compact maximal torus. Similarly, an L -packet of discrete series representations is parametrized by $\mathcal{E}(T)$.

For example, If we consider the connected group $G' = G'_{\text{sc}} = SU(2, 1)$ over \mathbb{R} , its K -group $G = G' \amalg G_1$, where $G_1 = SU(3)$. Then a stable conjugacy class of regular elliptic elements of G' consists of three conjugacy classes parametrized by three of the four elements of $H^1(\Gamma, T)$. Similarly, an L -packet of discrete series representations is parametrized by three elements of the same group, and we can obtain the fourth conjugacy class and a fourth representation from G_1 .

We also extend Langlands-Shelstad transfer mapping to K -groups.

$$\begin{aligned} \varphi : \mathcal{H}(G, \zeta) &\rightarrow SI(\widetilde{G}', \widetilde{\zeta}') \\ \varphi(f) = f'(\delta') &= \sum_{\gamma \in \Gamma_{\text{reg}}(G, \zeta)} \Delta(\delta', \gamma) f_G(\gamma). \end{aligned}$$

φ sends $\mathcal{H}(G, \zeta)$ continuously to the space $SI(\widetilde{G}', \widetilde{\zeta}')$. Here $\mathcal{H}(G, \zeta)$ is the Hecke algebra, $SI(\widetilde{G}', \widetilde{\zeta}')$ is the space of stable orbital integrals of functions. If F is non-archimedean, this is the main result of Waldspurger [33] and Ngo [20]. If F is Archimedean, the result was proved in Shelstad's paper [27].

If F is a global field, G is a K -group over F that satisfies the global analogue of the property as above, together with a local product structure, that is, for every frame (ψ, u) , the functions $u_{\alpha\beta} : \Gamma \rightarrow G_{\alpha, \text{sc}}$ are 1-cocycles for any α , the map $\{u_{\alpha\beta}\} \rightarrow H^1(F, G_\alpha)$ is a bijection onto the image of $H^1(F, G_{\alpha, \text{sc}})$ in $H^1(F, G_\alpha)$. We can write G as a product of

local K -groups, we define a local product structure on G to be a family of local K -groups (G_v, F_v) , indexed by the valuations of F , and a family of homomorphisms $G \rightarrow G_v$ over F_v whose restricted direct product $G(\mathbb{A}) \rightarrow \prod_v G_v(F_v)$ is an isomorphism over \mathbb{A} . Such a structure determines a surjective map

$$\alpha \mapsto \alpha_V = \prod_{v \in V} \alpha_v, \quad \alpha \in \pi_0(G), \alpha_v \in \pi_0(G_v),$$

of components, for any finite set V of places of F , which is bijective if V contains F_∞ . We also have a group theoretic injection of $G_\alpha(F)$ to $G_{\alpha_v}(F_v)$ for each $\alpha \in \pi_0(G)$. We shall write

$$G_V(F_V) = \prod_{v \in V} G_v(F_v) = \prod_{v \in V} \prod_{\alpha_v} G_{v, \alpha_v}(F_v) = \prod_{\alpha_V} G_{V, \alpha_V}(F_V).$$

Suppose that G is a K -group over F , and G^* is a quasisplit inner twist of G , then G^* is a connected quasisplit group over F , together with a inner class of inner twists $\psi_\alpha : G_\alpha \rightarrow G^*$ and a corresponding family of functions $u_\alpha : \Gamma \rightarrow G_{\text{sc}}^*$, for $\alpha \in \pi_0(G)$, then G^* determines a quasisplit inner twist G_v^* of each local K -group G_v . We shall refer to G as an inner K -form of G^* . As in the local case, we similarly construct the other basic objects.

If γ_V is an element in the set $\Gamma_{G_V}(M_V)$, let $\alpha_V \in \pi_0(M_V)$ be the index such that γ_V belongs to $\Gamma_{G_V}(M_{\alpha_V})$, we define the weighted orbital integral of f at γ_V simply by

$$J_{M_V}(\gamma_V, f_V) = J_{M_V}(\gamma_V, f_{\alpha_V}) \quad f_V \in \mathcal{H}(G_V, \zeta_V).$$

where $J_{M_V}(\gamma_V, f_{\alpha_V})$ is the weighted orbital integral on $G_{\alpha_V}(F_V)$. Similarly, we set

$$I_{M_V}(\gamma_V, f_V) = I_{M_V}(\gamma_V, f_{\alpha_V}) \quad f_V \in \mathcal{H}(G_V, \zeta_V),$$

where $I_{M_V}(\gamma_V, f_{\alpha_V})$ is the invariant distribution on $G_{\alpha_V}(F_V)$. $I_{M_V}(\gamma_V, f_{\alpha_V})$ is obtained from $J_{M_V}(\gamma_V, f_{\alpha_V})$ by adding some correction terms built out of weighted characters.

If $f_V = \oplus_{\alpha_V} f_{\alpha_V}$ in $\mathcal{H}(G_V, \zeta_V)$, $f_{G_V}(\gamma_V)$ denotes the invariant orbital integral $I_{G_V}(\gamma_V, f_{\alpha_V})$,

$$f'_V(\delta'_V) = f_V^{G'_V}(\delta'_V) = \sum_{\gamma_V \in \Gamma(G_V)} \Delta_{G_V}(\delta'_V, \gamma_V) f_{G_V}(\gamma_V), \quad \delta'_V \in \Delta_{G_V}(\tilde{G}'_V),$$

then $f'_V = \sum_{\alpha_V} f'_{\alpha_V}$. The base point of transfer factors for K -groups is that f'_V depends only on the base point $(\bar{\delta}'_V, \bar{\gamma}_V)$, where $\bar{\delta}'_V$ strongly G -regular, rather than a base point for each G_{α_V} . The Langlands-Shelstad transfer Theorem, applied to each of the groups G_{α_V} , asserts that f'_V belongs to the space $SI\mathcal{H}(\tilde{G}'_V, \tilde{\zeta}'_V)$ of stable orbital integrals of functions in $\mathcal{H}(\tilde{G}'_V, \tilde{\zeta}'_V)$, where \tilde{G}'_V comes with a central data $(\tilde{Z}'_V, \tilde{\zeta}'_V)$.

Similarly, we can define the objects on K -group on the spectral side. I will stabilize the spectral side of the local trace formula for the K -group over \mathbb{R} in section 5, section 6, section 7.

3. MULTIPLICITY OF DISCRETE SERIES AND INVARIANT TRACE FORMULA

Suppose that F is a number field, G is a connected reductive K -group over F . We can form the adélic ring $\mathbb{A}_F = \prod'_v F_v$, and the group of adélic points of G is $G(\mathbb{A}_F) = \prod'_v G(F_v)$.

Automorphic representations of G over F are irreducible constituents of the right regular representation of R , by which we mean is that the right action defined by

$$R(x)\varphi(y) = \varphi(yx), \quad \varphi \in L^2(G(F)\backslash G(\mathbb{A}_F)).$$

The fundamental problem in the automorphic representation theory is to decompose R . It is well known that R can be decomposed into a discrete spectrum and a continuous spectrum,

$$R = R_{\text{disc}} \oplus R_{\text{cont}}.$$

Langlands [23] has studied the continuous spectrum by Eisenstein series. So it remains to study the discrete part

$$(3.1) \quad R_{\text{disc}} = \bigoplus_{\pi \in \Pi_{\text{disc}}(G(\mathbb{A}))} m_{\text{disc}}(\pi) \pi$$

where $\Pi_{\text{disc}}(G(\mathbb{A}))$ stands for the set of equivalence classes of irreducible representations of a given group $G(\mathbb{A})$ on $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}_F))$.

Suppose π is an automorphic representation of $G(\mathbb{A}_F)$, we have a decomposition

$$\pi = \bigotimes_v \pi_v,$$

such that

- (1) π_v is an irreducible admissible representation of $G(F_v)$;
- (2) π_v is unramified for almost all v .

We will study the real component of π . We denote $\pi = \pi_{\mathbb{R}} \otimes \pi_{\text{fin}}$, where $\pi_{\mathbb{R}}$ and π_{fin} are the irreducible representations of $G(\mathbb{R})$ and $G(\mathbb{A}_{\text{fin}})$ respectively. To compute the global multiplicity $m_{\text{disc}}(\pi)$ of G is the recent work of Arthur about the classical group and Mok about the unitary group. I will study the multiplicity formula of $m_{\text{disc}}(\pi_{\mathbb{R}})$ for connected K -group over \mathbb{Q} , in a different way. Our original problem comes from [3, p,284], Arthur applied the invariant trace formula to compute the L^2 -Lefschetz numbers of Hecke operators. At the same time, he obtained the sum of multiplicity formula for $\sum_{\pi \in \Pi_{\text{disc}}(\mu)} m_{\text{disc}}(\pi_{\mathbb{R}}, K_0)$, under a weak regularity assumption on the representations in $\Pi_{\text{disc}}(\mu)$, when G is connected reductive group, and the packet $\Pi_{\text{disc}}(\mu)$ consists of the set of discrete series representations with the same infinitesimal character μ . This paper will give a formula for single multiplicity $m_{\text{disc}}(\pi_{\mathbb{R}}, K_0)$ with which $\pi_{\mathbb{R}}$ occurs discretely in the representation of $G(\mathbb{R})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_0, \zeta)$. Since the double coset space

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})K_0$$

is finite, we denote by $x_1 = 1, x_2 \cdots x_n$ the set of representatives in $G(\mathbb{A}_{\text{fin}})$, the groups $\Gamma_i = (G(\mathbb{Q}) \cdot x_i K_0 x_i^{-1}) \cap G(\mathbb{R})$, $1 \leq i \leq n$ are arithmetic subgroups of $G(\mathbb{R})$, we obtain that $G(\mathbb{Q})\backslash G(\mathbb{A})/K_0$ is the disjoint union of space $\Gamma_i \backslash G(\mathbb{R})$. This fact have built the bridge of local and group information.

The question of the multiplicity is quite natural from the point of view of spectral theory, More generally, one can consider Hecke operators h on $L^2(G(F)\backslash G(\mathbb{A}))$, where h is a K_0 -bi-invariant function in $\mathcal{H}(\mathbb{A}_{\text{fin}})$. Any such operator commutes with the action of $G(\mathbb{R})$. It restricts to the subspace $R_{\text{disc}}(\pi_{\mathbb{R}})$ by denoted $R_{\text{disc}}(\pi_{\mathbb{R}}, h)$, this paper is also to give a formula for $\text{tr}(R_{\text{disc}}(\pi_{\mathbb{R}}, h))$, under same regularity condition.

If G is a connected reductive group over \mathbb{Q} , the multiplicities of discrete series have a homological interpretation. The global multiplicity $m_{\text{disc}}(\pi)$ occurs in the well known isomorphism

$$H_{(2)}^q(h, \mathcal{F}_\mu) \cong \bigoplus_{\pi \in \Pi(G(\mathbb{A}), \zeta)} (m_{\text{disc}}(\pi) \dim H^q(\mathfrak{g}(\mathbb{R}), K_{\mathbb{R}}; \pi_{\mathbb{R}} \otimes \mu)) \pi_{\text{fin}}(h).$$

(see [3, §2], the coefficient $m_{\text{disc}}(\pi)$ stands for the multiplicity of the ensuing space in the direct sum) where \mathfrak{g} is the Lie algebra of G ; Both of $\mathfrak{g}(\mathbb{R})$ and $K_{\mathbb{R}}$ act on the space of $K_{\mathbb{R}}$ -finite vectors of the representation $\pi_{\mathbb{R}} \otimes \mu$ of $G(\mathbb{R})$. The relative Lie algebra cohomology groups $H^q(\mathfrak{g}(\mathbb{R}), K_{\mathbb{R}}; \pi_{\mathbb{R}} \otimes \mu)$ give the contribution of $\pi_{\mathbb{R}}$ to the cohomology. $V(\pi_{\text{fin}}^{K_0})$ denotes the subspace of vectors in the underlying space of π_{fin} which are fixed by K_0 , this is a finite dimensional subspace which gives the contribution of π_{fin} to the cohomology. If the multiplicity of $\pi_{\mathbb{R}}$ occurs discretely in the representation of $G(\mathbb{R})$ on $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A})/K_0, \zeta)$, then

$$(3.2) \quad m_{\text{disc}}(\pi_{\mathbb{R}}) = \sum_{\substack{\pi = \pi_{\mathbb{R}} \otimes \pi_{\text{fin}} \\ \pi \in \Pi(G(\mathbb{A}), \zeta)}} m_{\text{disc}}(\pi) \dim(V(\pi_{\text{fin}}^{K_0})).$$

A representation of K -group G is determined by the representations of connected components of G . If $\pi_\alpha \in \Pi(G_\alpha, \zeta)$, $f_\alpha \in C(G_\alpha, \zeta)$, $f \in C(G, \zeta)$, then we define

$$f_G(\pi_\alpha) = f_{G_\alpha}(\pi_\alpha), \quad f = \bigoplus_\alpha f_\alpha, \quad \alpha \in \pi_0(G).$$

So we can naturally extend the homological interpretation for the representation on the connected component of K -group to the K -group, then we can extend (3.2) to K -group.

We write $\Pi_2(G(\mathbb{R}), \zeta)$ and $\Pi_{\text{temp}}(G(\mathbb{R}), \zeta)$ for the set of discrete series and the set of tempered representations, whose central character coincide with a given character ζ on Z respectively. $C(G(\mathbb{R}), \zeta)$ is the Schwartz space, whose central character is given by a character ζ^{-1} on Z .

If f is any function in $C(G(\mathbb{R}), \zeta)$ and $\pi_{\mathbb{R}}$ belongs to $\Pi(G(\mathbb{R}), \zeta)$, we can set

$$\pi_{\mathbb{R}}(f) = \int_{G(\mathbb{R})/Z} f(x) \pi_{\mathbb{R}}(x) dx$$

Lemma 3.1. *There is a function $f_{\pi_\mu} \in C(G(\mathbb{R}), \zeta)$, such that for any $\pi \in \Pi_{\text{temp}}(G(\mathbb{R}), \zeta)$,*

$$\text{tr}(\pi_{\mathbb{R}}(f_{\pi_\mu})) = \begin{cases} 1 & \text{if } \pi_{\mathbb{R}} = \pi_\mu, \pi_\mu \in \Pi_{\text{disc}}(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

The above lemma is an immediate consequence of the trace Paley-Wiener theorem of Arthur [6]. We shall call f_{π_μ} a pseudo-coefficient of π_μ , and we say that f is cuspidal. if $\text{tr } \pi(f)$ viewed as function on $\Pi_{\text{temp}}(G(\mathbb{R}), \zeta)$ is supported on $\Pi_2(G(\mathbb{R}), \zeta)$. So f_{π_μ} is cuspidal.

Now, we fix the function $h \in \mathcal{H}(G(\mathbb{A}_{\text{fin}}))$, and set $(f_{\pi_\mu} h)(x) = f_{\pi_\mu}(x_{\mathbb{R}}) h(x_{\text{fin}})$, $x = x_R x_{\text{fin}}$, $x_R \in G(\mathbb{R})$, $x_{\text{fin}} \in G(\mathbb{A}_{\text{fin}})$ in $G(\mathbb{A})$. If $\pi = \pi_R \otimes \pi_{\text{fin}}$ is any representation in

$\Pi(G(\mathbb{A}, \zeta))$, we have

$$\begin{aligned} \mathrm{tr} \pi(f_{\pi_\mu} h) &= \mathrm{tr} \left(\int_{G(\mathbb{A})/Z} (f_{\pi_\mu} h)(x) \pi(x) dx \right) \\ &= \mathrm{tr} \pi_{\mathbb{R}}(f_{\pi_\mu}) \mathrm{tr} \pi_{\mathrm{fin}}(h). \end{aligned}$$

Since f_{π_μ} is cuspidal, which will cancel the contributions from Levi subgroups, so the invariant trace formula is simple. The spectral side of the trace formula is

$$\begin{aligned} I(f_{\pi_\mu} h) &= \sum_{\pi \in \Pi(G(\mathbb{A}), \zeta)} m_{\mathrm{disc}}(\pi) \mathrm{tr} \pi(f_{\pi_\mu} h) \\ &= \sum_{\pi \in \Pi(G(\mathbb{A}), \zeta)} m_{\mathrm{disc}}(\pi) \mathrm{tr} \pi_{\mathbb{R}}(f_{\pi_\mu}) \mathrm{tr} \pi_{\mathrm{fin}}(h) \\ &= \sum_{\pi = \pi_{\mathbb{R}} \otimes \pi_{\mathrm{fin}}} m_{\mathrm{disc}}(\pi) \mathrm{tr} \pi_{\mathrm{fin}}(h). \end{aligned}$$

If we take h to equal the unit I_{k_0} in $\mathcal{H}(\mathbb{A}_{\mathrm{fin}})$, then $\mathrm{tr} \pi_{\mathrm{fin}}(h) = \dim(V(\pi_{\mathrm{fin}}^{k_0}))$. So

$$(3.3) \quad I(f_{\pi_\mu} I_{k_0}) = m_{\mathrm{disc}}(\pi_\mu, K_0),$$

and

$$(3.4) \quad I(f_{\pi_\mu} h) = \mathrm{tr}(R_{\mathrm{disc}}(\pi_\mu, h)).$$

We will expand $m_{\mathrm{disc}}(\pi_\mu)$ for the geometric side of invariant trace formula, but we can not obtain explicit formula from the invariant trace formula. f_{π_μ} is cuspidal but it is not stable function, so we can not cancel the contribution of the unipotent part distributions in the invariant trace formula.

If the test function f is stable cuspidal, we can obtain an explicit formula though the invariant trace formula [3]. We can easily extend the results to K -group from section 2. If G is a connected K -group over \mathbb{R} , $f \in C(G(\mathbb{R}), \zeta)$ is stable cuspidal and $\gamma \in M(\mathbb{R})$, where $M(\mathbb{R})$ is a Levi subgroup of $G(\mathbb{R})$, then

$$(3.5) \quad \Phi_M(\gamma, f) = (-1)^{\dim(A_M/A_G)} v(M_\gamma)^{-1} \sum_{\tau \in \Pi(G^*(\mathbb{R}), \zeta)} \Phi_M(\gamma, \tau) \mathrm{tr} \tilde{\tau}(f).$$

In particular, $\Phi_M(\gamma, f)$ vanishes if γ is not semisimple (see [3, Theorem 5.1]), where

$$\begin{aligned} \Phi_M(\gamma, f) &= |D^M(\gamma)|^{-1/2} I_M(\gamma, f), \quad v(G) = (-1)^{q(G)} \mathrm{vol}(G^*/A_G(\mathbb{R})^\circ) |\mathcal{D}(G, B)|^{-1} \\ \Phi_M(\gamma, \tau) &= (-1)^{q(G)} |D_M^G(\gamma)|^{1/2} \sum_{\pi \in \Pi_{\mathrm{disc}}(\tau)} \Theta_\pi(\gamma), \end{aligned}$$

where $\Pi_{\mathrm{disc}}(\tau)$ is just a L -packet of discrete series, $q(G) = \frac{1}{2} \dim(G(\mathbb{R})/K_{\mathbb{R}} A_G(\mathbb{R}))$, $K_{\mathbb{R}}$ is a maximal compact subgroup of $G(\mathbb{R})$. The invariant distribution $\Phi_M(\gamma, \tau)$ follows from Harish-Chandra's stable character of discrete series, which is interesting. Let us review the explicit formula for the averaged discrete series characters. We can naturally extend it to the K -group $G(\mathbb{R})$. We suppose the group $G(\mathbb{R})$ is a connected reductive Lie group. Let $Z(B)$ be the centralizer of the connected component $G(\mathbb{R})^\circ$ in $K_{\mathbb{R}}$. $B(\mathbb{R})$ [13, Lemma 3.4]

equals the product of its connected component $B(\mathbb{R})^\circ$ with $Z(B)$. We set ρ_B as usual to be half of the sum of positive roots of (G, B) . Let $\Lambda(\zeta)$ denote the set of pairs

$$(\zeta, \lambda), \quad \zeta \in Z(B)^*, \lambda \in b(\mathbb{C})^*,$$

such that $z \exp H \rightarrow \zeta(z)e^{(\lambda - \rho_B)(H)}$, $Z \in Z(B)$, $H \in b(\mathbb{R})$ is a well defined quasi-character on $B(\mathbb{R})$ whose restriction to $A_G(\mathbb{R})^\circ$ equals ζ , and λ is regular. $\Lambda(\zeta)$ equipped with an action of Weyl group $W(G, B)$. The discrete series are parameterized by the $W(G(\mathbb{R}), B(\mathbb{R}))$ -orbits in $\Lambda(\zeta)$. We denote by ϕ a discrete Langlands parameter, then we can find that a L -packet Π_ϕ corresponds to the partition of a given $W(G, B)$ -orbit into $W(G(\mathbb{R}), B(\mathbb{R}))$ -orbits [27], we set $\mathcal{D}(G, B) = W(G, B)/W(G(\mathbb{R}), B(\mathbb{R}))$, then

$$|\mathcal{D}(T)| = |\mathcal{D}(G, B)|,$$

where T is a maximal torus which is \mathbb{R} -anisotropic modulo $A(\mathbb{R})^\circ$, T is conjugate with B .

Assume that G^* is connected reductive group over \mathbb{R} , $\eta : G^* \rightarrow G$ is an isomorphism over \mathbb{C} such that the automorphism $\eta^\sigma \eta^{-1}$ is inner for $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$. We use η to identify A_G with the R -split component of the center of G^* , and we assume that $G^*(\mathbb{R})/A_G(\mathbb{R})$ is compact. Then the representations in $\pi(G^*)$ are all finite dimensional, According to Langlands classification [22], the set $\Pi_{\text{disc}}(G(\mathbb{R}))$ is a disjoint union of finite subsets $\Pi_{\text{disc}}(\tau)$, which are parametrized by the irreducible representation τ in $\Pi(G^*(\mathbb{R}))$. If τ and ϕ are parametrized by the same infinitesimal character, then the L -packet Π_ϕ equal the finite set $\Pi_{\text{disc}}(\tau)$. We can set $\Phi_M(\gamma, \phi) = \Phi_M(\gamma, \tau)$.

For given $\tau \in \Pi(G^*(\mathbb{R}), \zeta)$, let $(\zeta, \lambda) \in \Lambda(\zeta)$ be the point in the corresponding orbit, such that λ is positive on all the positive co-roots of (G, B) , then if

$$\gamma = z \exp H, z \in Z(B), H \in b(\mathbb{R})$$

is a regular point in $B(\mathbb{R})$, we have

$$\Phi_G(\gamma, \tau) = \text{tr } \tau(\gamma) = \Delta_B^G(H)^{-1} \zeta(z) \sum_{s \in W(G, B)} \varepsilon(s) e^{(s\lambda)(H)}.$$

Here $\Delta_B^G(H) = \prod_{\alpha > 0} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)})$.

For the general averaged discrete series character $\Phi_M(\gamma, \tau)$, let T be a maximal torus in M which is \mathbb{R} -anisotropic modulo $A_M(\mathbb{R})^\circ$. R is the set of real roots of (G, T) . The existence of torus B means that $W(\mathbb{R})$ contains an element that acts as -1 . We can take T from its $M(\mathbb{R})$ -conjugates so that $T = (T \cap B)A_M$. Then there is an element $y \in G(\mathbb{C})$, such that $\text{Ad}(y)(b(\mathbb{C})) = t(\mathbb{C})$, where t is the Lie algebra of T .

Suppose that $\tau \in \Pi(G(\mathbb{R}), \xi)$. $(\xi, \lambda) \in \Lambda(\xi)$ is a point in the corresponding $W(G, B)$ -orbit such that $y\lambda$ is positive on all positive co-roots of (G, T) . Then $\Phi_M(\gamma, \tau)$ vanishes for any regular point $\gamma \in T(\mathbb{R})$ unless γ is of the form

$$\gamma = z \exp(H), \quad z \in Z(B), H \in t(\mathbb{R}),$$

in which case

$$(3.6) \quad \Phi_M(\gamma, \tau) = \Delta_T^M(H)^{-1} \varepsilon_R(H) \xi(z) \sum_{s \in W(G, B)} \varepsilon(s) \bar{C}(Q_{ys\lambda}^+, R_H^+) e^{(ys\lambda)(H)}.$$

Here $\varepsilon_R(H) = (-1)^{|R_H^+ \cap (-R^+)|}$, H is a regular point in $t(\mathbb{R})$, which is the Lie algebra of $T(\mathbb{R})$, and R_H^+ for the set of roots which are positive on H , $\varepsilon(s)$ is a sign function on $W(G, B)$, and $\bar{C}(Q^+, R^+)$ is an integer valued functions, which defined for root systems R whose Weyl group $W(\mathbb{R})$ contains (-1) . The function $\bar{C}(Q^+, R^+)$ are uniquely determined by the following four properties.

- (1) $\bar{C}(sQ^+, sR^+) = \bar{C}(Q^+, R^+)$, $s \in W(R)$.
- (2) The number $\bar{C}(Q^+, R^+)$ vanishes unless $\nu(X)$ negative for every $X \in a_{R^+}$, and $\nu \in a_{Q^+}$.
- (3) $\bar{C}(Q^+, R^+) + \bar{C}(s_\alpha Q^+, R^+) = 2\bar{C}(Q^+ \cap Q_\alpha, R^+ \cap R_\alpha)$, for any reflection $s_\alpha \in W(R)$ corresponding to a root $\alpha \in R$.
- (4) If R is the empty root system, then $\bar{C}(Q^+, R^+) = 1$.

Here R^+ is a system of positive roots for R , and Q^+ is a positive system for the set $Q = R^\vee$ of co-roots, Q_λ^+ is the set of co-roots α^\vee of $Q = R^\vee$ for which $\lambda(\alpha^\vee)$ are all positive.

Arthur [3] extends $\Phi(\gamma, \tau)$ to a continuous, $W(M, T)$ -invariant function on $T(\mathbb{R})$, then extends to a function on $M(\mathbb{R})$, which is a connected reductive group, and $\Phi(\cdot, \tau)$ is a $M(\mathbb{R})$ -invariant function on $M(\mathbb{R})$ which is supported on the $M(\mathbb{R})$ -elliptic conjugacy classes. If $M \in \mathcal{L}$ is not cuspidal, then we set $\Phi_M(\gamma, \tau)$ vanishes.

If the representative of the conjugacy class $\gamma \in (M(\mathbb{Q}))_{M,S}$ is semisimple, it is independent of S . Moreover, for any semisimple element $\gamma \in M(\mathbb{Q})$,

$$a^M(S, \gamma) = |\iota^M(\gamma)|^{-1} \text{vol}(M_\gamma(\mathbb{Q}) \backslash M_\gamma(\mathbb{A})^1)$$

If γ is not \mathbb{Q} elliptic in M , then $a^M(S, \gamma)$ vanishes, where $|\iota^M(\gamma)| = |M_\gamma(\mathbb{Q}) \backslash M(\mathbb{Q}, \gamma)|$, $M_\gamma(\mathbb{A})^1 = A_M(\mathbb{R})^0 \backslash M_\gamma(\mathbb{A})$. So if f is stable cuspidal, we have an explicit geometric expansion of general global trace formula,

$$(3.7) \quad I(fh) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))_{M,S}} a^M(S, \gamma) I_M(\gamma, fh),$$

$$\text{and } I_M(\gamma, fh) = I_M^G(\gamma, f) I_M^M(\gamma, h) = |D^M(\gamma)| \Phi_M(\gamma, f) h_M(\gamma).$$

4. GLOBAL STABLE TRACE FORMULA

Suppose G is a K -group over a number field F , the invariant trace formula can be extended to K -group, which is stabilized by Arthur in [9], [10], [11]. We need to recall the basic information about the stable trace formula before applying it to obtain the multiplicity formula. Let S be a finite set of valuations of F that contains the set of places at which G ramifies, the general trace formula is the identity obtained from two different expansions of a certain linear form $I(f)$ in [2], $f \in \mathcal{H}(G, S)$ on the Hecke algebra of $G(F_S)$, the geometric expansion,

$$(4.1) \quad I(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in \Gamma(M, S)} a^M(S, \gamma) I_M(\gamma, f)$$

is a linear combination of distributions parametrized by conjugacy classes γ in Levi subgroups $M(F_S)$. The spectral expansion

$$(4.2) \quad I(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{t > 0} \int_{\Pi_t(M, S)} a^M(S, \pi) I_M(\pi, f) d\pi$$

is a linear combination of distributions parametrized by representations π of Levi subgroup $M(F_S)$.

Arthur concerned the more general coefficients $a^M(\gamma)$ and $a^M(\pi)$, which are the global objects and contain almost all (the meaning that is a large finite set of places S and independent of the S) the unramified weighted orbit integral and weighted character respectively. $I_M(\gamma, f)$ and $I_M(\pi, f)$ are the local object, we can apply splitting formula and descent formula to reduce the distribution to Levi subgroup of G . The multiplicity formula results are described by the geometric side of the global trace formula. So we will just concern the geometric side of the global trace formula.

We need to stabilize the coefficient $a^M(\gamma)$ and the invariant distribution $I_M(\gamma, f)$. Those are the corresponding global theorem and local theorem [11]. Firstly, we recall the relation of the coefficient $a^M(\gamma)$ and $a^M(S, \gamma)$. We will freely use the notation in [9], the original coefficient $a^G(S, \gamma)$ in (3.7) is defined on $(G(F))_{G, S}$, where $(G(F))_{G, S}$ is the set of what are called the (G, S) -equivalence classes in $G(F)$, the two elements γ and γ_1 in $G(F)$, with standard Jordan decompositions $\gamma = c\alpha$ and $\gamma_1 = c_1\alpha_1$, were defined to be (G, S) -equivalent if there is an element $\delta \in G(F)$ such that $\delta^{-1}c_1\delta = c$, and such that $\delta^{-1}\alpha_1\delta$ is conjugate to α in $G_c(F_S)$. If γ is semisimple, then it corresponds to the conjugacy class as usual.

For a general element $\gamma = c\alpha$, c is the semisimple part, α is the unipotent part, the coefficient is defined by a descent formula,

$$a^G(S, \gamma) = i^G(S, c) |\text{stab}(c, \alpha)|^{-1} a^{G_c}(S, \alpha),$$

where $\text{stab}(c, \alpha)$ stands for the stabilizer of α in the finite group $(G_{c,+}(F)/G_c(F))$, which acts on the set of unipotent conjugacy classes in $G_c(F_S)$. $i^G(S, c)$ is equal to 1, if c is F -elliptic in G , and the $G(\mathbb{A}^S)$ conjugacy class of c meets K^S , and is otherwise equal to 0. The descent formula is to reduce the study of $a^G(S, \gamma)$ to the case of unipotent elements. In fact, the coefficient $a^G(S, \gamma)$ is defined for any finite set S , and we can transfer $a^G(S, \gamma)$ on the classes in $(G(F))_{G, S}$ to the abstract basis $\Gamma(G_S^Z, \zeta_S)$, where $G^Z = \{x \in G : H_G(x) \in \text{Image}(\mathbf{a}_Z \rightarrow \mathbf{a}_G)\}$.

We set

$$a_{\text{ell}}^G(\gamma) = \sum_{\{\gamma\}} |Z(F, \gamma)|^{-1} a^G(S, \gamma) (\gamma_S / \gamma)^{-1},$$

where $\{\gamma\}$ is summed over those $Z_{S, O} = Z(F) \cap Z_S Z(O)^S$ orbit in $(G(F))_{G, S}$ that map to γ_S , and such that the $G(\mathbb{A}^S)$ conjugacy class of γ in $G(\mathbb{A}^S)$ meets K^S , $Z(F, \gamma) = \{z \in Z(F) : z\gamma = \gamma\} = \{z \in Z_{S, O} : z\gamma = \gamma\}$, and γ_S / γ is the ratio of the invariant measure on γ_S and the signed measure on γ_S that comes with γ . The coefficient $a_{\text{ell}}^G(\gamma)$ is supported on the set of admissible elements in the discrete subset $\Gamma_{\text{ell}}(G, S, \zeta)$ (see [9, (2.6)]) of $\Gamma(G_S^Z, \zeta_V)$, where admissible elements is defined in [9, §1].

If M is a Levi subgroup of G , and μ belongs to $\Gamma(M_S^Z, \zeta_S)$, the induced distribution μ^G is a finite linear combination of elements in $\Gamma(G_S^Z, \zeta_S)$. We write $\Gamma(G, S, \zeta)$ for the set of elements so obtained, as M ranges over \mathcal{L} and μ runs over the element $\Gamma_{\text{ell}}(M, S, \zeta)$, these objects are compatible with the spectral side.

If γ belongs to $\Gamma(G_V^Z, \zeta_V)$, $V_{\text{ram}} \subset V \subset S$, we denote [9, (2.8)]

$$a^G(\gamma) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{k \in \mathcal{K}_{\text{ell}}^V(M/Z, S)} a_{\text{ell}}^M(\gamma_M \times k) r_M^G(k),$$

where $r_M^G(k) = \mathcal{J}_M(r_S^V(k), u_S^V)$, $k \in \mathcal{K}((M/Z)_S^V)$ is the unramified weighted orbital integrals, $\mathcal{K}_{\text{ell}}^V(G/Z, S)$ for the set of k in $\mathcal{K}((G/Z)_S^V)$ such that $\gamma \times k$ belongs to $\Gamma_{\text{ell}}(G, S, \zeta)$ for some γ .

If $f \in \mathcal{H}(G, V, \zeta)$, then the linear form $I(f)$ has a geometric expansion [9, Proposition 2.2]

$$(4.3) \quad I(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in \Gamma(M, V, \zeta)} a^M(\gamma) I_M(\gamma, f)$$

The general trace formula was stabilized by Arthur, who inductively defined a general stable distribution on the quasisplit group G' which is independent of the G , and then built up an endoscopy trace formula.

If G is quasisplit, Arthur [11] proved

$$S^G(f) = I(f) - \sum_{G' \in \mathcal{E}_{\text{ell}}^0(G)} \iota(G, G') \widehat{S}^{G'}(f')$$

is stable. If G is general, he [11] proved

$$I(f) = I^{\mathcal{E}}(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \widehat{S}^{G'}(f').$$

The endoscopic trace formula $I^{\mathcal{E}}(f)$ and the stable trace formula $S^G(f)$ both have a geometric expansion

$$(4.4) \quad I^{\mathcal{E}}(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in \Gamma^{\mathcal{E}}(M, V, \zeta)} a^{M, \mathcal{E}}(\gamma) I_M^{\mathcal{E}}(\gamma, f),$$

and

$$(4.5) \quad S^G(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{\delta \in \Delta(M, V, \zeta)} b^M(\delta) S_M^G(\delta, f).$$

Here the coefficient relation is

$$a^G(\gamma) = a^{G, \mathcal{E}}(\gamma) = \sum_{G'} \sum_{\delta'} \iota(G, G') b^{\widetilde{G}'}(\delta') \Delta_G(\delta', \gamma) + \varepsilon(G) \sum_{\delta} b^G(\delta) \Delta_G(\delta, \gamma).$$

with $\gamma \in \Gamma(G_V^Z, \zeta_V)$, G', δ' and δ summed over $\mathcal{E}_{\text{ell}}^0(G)$, $\Delta((\widetilde{G}'_V)^{Z'}, \widetilde{\zeta}'_V)$ and $\Delta^{\mathcal{E}}(G_V^Z, \zeta_V)$ respectively. The stable coefficient $b^G(\delta)$ is stable, the meaning that is $b^G(\delta)$ is supported on $\Delta(G, V, \zeta)$, and the coefficients $a^G(\gamma)$ and $b^G(\delta)$ are independent of S .

To stabilize the general trace formula which essentially amounts to compare the two expansions (4.4) and (4.5). Assuming the fundamental Lemma, Arthur [11] have proved this, and in 2008 Ngo [20] proved the fundamental Lemma. So we now have obtained the unconditional stable trace formula.

We have a formula about the stable coefficient

$$b^G(\delta) = b_{\text{ell}}^G(\delta, S) + \sum_{M \in \mathcal{L}^0} |W_0^M| |W_0^G|^{-1} \sum_{k \in \mathcal{K}_{\text{ell}}^V(M/Z, S)} b_{\text{ell}}^M(\delta_M \times k) S_M^G(k),$$

where the stable term $S_M^G(k)$ comes from the unramified weighted orbital integrals $r_M^G(k)$ (fundamental Lemma applies to this part). Here $b_{\text{ell}}^G(\delta)$ also have the descent formula [10], if δ_S is an admissible element in $\Delta_{\text{ell}}(G, S, \zeta)$ with Jordan decomposition $\delta_S = d_S \beta_S$, then

$$b_{\text{ell}}^G(\delta_S) = \sum_d \sum_{\beta} j^{G^*}(S, d) b_{\text{ell}}^{G_d^*}(\beta),$$

where d is summed over the set of elements in $\Delta_{\text{ss}}(G^*)$ whose image in $\Delta_{\text{ss}}(G_S^*)$ equal d_S , where G^* is quasisplit inner form of G , and β is summed over the orbit of $(\overline{G^*}/\overline{G^*_{d,S}})(F)$ in $\Delta_{\text{unip}}(G_{d,S}^*, \zeta)$. Moreover, b_{ell}^G vanishes on the complement of $\Delta_{\text{ell}}(G, S, \zeta)$ in the set of admissible elements in $\Delta_{\text{ell}}^{\mathcal{E}}(G, S, \zeta)$, whose semisimple part is not central in G_S^* (see [10, Theorem 1.1]), and $j^{G'}(S, d') = i^{G'}(S, d') \tau(G') \tau(G'_{d'})^{-1}$ (see [10, (1.7)]).

If δ is semisimple, elliptic in $\Delta_{\text{ell}}(G', V, \zeta)$, by [17, Theorem 8.3.1], or [11, page 105], then

$$\begin{aligned} b^{G'}(\delta') &= b_{\text{ell}}^{G'}(\delta') = j^{G'}(S, \delta') b_{\text{ell}}^{G'}(1) \\ &= \tau(G') \tau(T)^{-1} \tau(T) = \tau(G') \end{aligned}$$

where $T = G'_{d'}$ and $\tau(G') = |\pi_0(Z(\hat{G}')^\Gamma)| |\text{Ker}^1(F, Z(\hat{G}'))|^{-1}$ is Tamagawa number of G' [16, (5.1.1)], [18].

we obtain a stable form of the invariant trace formula

$$I(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \sum_{M' \in \mathcal{L}^{G'}} |W_0^{M'}| |W_0^{G'}|^{-1} \sum_{\delta \in \Delta(M', V, \zeta)} b^{M'}(\delta) S_{M'}^{G'}(\delta, f).$$

Here $\Delta(M', V, \zeta)$ is the basis of $\Delta(M_V'^Z, \zeta)$, $\Delta(M_V'^Z, \zeta) = \Delta^{\mathcal{E}}(M_V'^Z, \zeta) \cap SD(M_V'^Z, \zeta)$, and $\Delta^{\mathcal{E}}(M_V'^Z, \zeta)$ as a quotient of the subset of M' relevant pairs in $\{(M_V''^Z, \delta') : M_V'' \in \mathcal{E}(M_V'), \delta' \in \Delta(M_V''^Z, \zeta)\}$, $\iota(G, G')$ is the Langlands's global coefficients,

$$\iota(G, G') = \iota(G_\alpha, G') = \tau(G) \tau(G')^{-1} |\text{Out}_G(G')|^{-1},$$

where $\alpha \in \pi_0(G)$, $G' \in \mathcal{E}_{\text{ell}}(G)$, $\text{Out}_G(G') = \text{Aut}_G(G') / \widehat{G'}$.

If $f = f_\infty h$, f_∞ is cuspidal in $C(G(\mathbb{R}), \zeta)$ and $h \in \mathcal{H}(G(\mathbb{A}_{\text{fin}}))$, then the term

$$S_M^G(\delta, f) = S_M^G(\delta, f_\infty) (f_V^\infty)^M(\delta), \quad \delta \in \Delta(M, V, \zeta).$$

This is a consequence of the stable splitting formula [8, Theorem 6.1]. We set $V = V_1 \amalg V_2$, $V_1 \supset \{\infty\}$, $f = f_{V_1} f_{V_2}$, then

$$S_M^G(\delta, f) = \sum_{L_1, L_2 \in \mathcal{L}(M)} e_M^G(L_1, L_2) \widehat{S}_M^{L_1}(\delta, f_{V_1, L_1}) \widehat{S}_M^{L_2}(\delta, f_{V_2, L_2}).$$

Where $e_M^G(L_1, L_2)$ was defined [8, Theorem 6.1]. Since f_∞ is cuspidal, so f_{V_1} is cuspidal, then $(f_{V_1})_{L_1} = 0$ for $L_1 \neq G$. On the other hand, $e_M^G(G, L_2) \neq 0$ only when $L_2 = M$ in which it is equal to 1. Then

$$S_M^G(\delta, f) = \widehat{S}_M^G(\delta, f_{V_1, G}) \widehat{S}_M^M(\delta, f_{V_2, M}).$$

However V is finite, so we continue this process to obtain

$$S_M^G(\delta, f_\infty h) = S_M^G(\delta, f_\infty) (h_V^\infty)^M(\delta).$$

We can get the following proposition by combining (3.4).

Proposition 4.1. *For any $h \in \mathcal{H}(G(\mathbb{A}_{\text{fin}}))$, $f_{\pi_\mu} \in C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, we have*

(4.6)

$$\begin{aligned} \text{tr}(R_{\text{disc}}(\pi_\mu, h)) &= I(f_{\pi_\mu} h) \\ &= \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \sum_{M' \in \mathcal{L}^{G'}} |W_0^{M'}| |W_0^{G'}|^{-1} \sum_{\delta \in \Delta(M', V, \zeta)} b^{M'}(\delta) S_{M'}^{G'}(\delta, f_{\pi_\mu}) (h_M)^{M'}(\delta) \end{aligned}$$

We now have a stable trace formula. In order to make this more explicit, we need to examine the individual terms more closely. The stable orbital integral $(h_M)^{M'}(\delta)$ is simple enough. It is based on transfer theorem. $S_M^G(\delta, f_{\pi_\mu})$ is a stable distribution attached to a invariant distribution, which is more complicated. We need to give an explicit formula for $S_M^G(\delta, f_{\pi_\mu})$. We know that the local component of global stable trace formula is more complicated than the local component of stable local trace formula. But if the test function is stable cuspidal, they are the same, because the component of invariant local trace formula and the local component of invariant trace formula is the same in [3], when G is a connected reductive group. However We have the following proposition.

Proposition 4.2. *If $G(\mathbb{R})$ is a K -group, G' is the endoscopy group of G , f is a cuspidal function in $C(G(\mathbb{R}), \zeta)$, $\delta \in M(\mathbb{R})$ is any element with Jordan decomposition $\delta = cu$, and u is not trivial unipotent element, then $S_{M'}^{G'}(\delta, f) = 0$.*

Proof. G is K -group over R , we can write $G = \coprod_{\alpha \in \pi_0(G)} (G_\alpha)$, where G_α is a connected reductive group. If $\delta \in M'_\alpha$ and $\delta = cu$, u is not trivial, then we have

$$S_{M'}^{G'}(\delta, f) = S_{M'_\alpha}^{G'_\alpha}(\delta, f_\alpha) = \widehat{S}_{M'_\alpha}^{G'_\alpha}(\delta, (f)^{G'_\alpha}) = 0.$$

The third equality is from Arthur's result [3, Theorem 5.1], We know that when G is a connected reductive group, the test function is stable cuspidal, then the invariant distribution vanishes except the semisimple part. However $f^{G'_\alpha}(\delta)$ is stable cuspidal function of G'_α by Shelstad transfer theorem, and the set of stable distributions are contained in the set of invariant distributions, so we obtain $S_{M'}^{G'}(\delta, f) = 0$. \square

From the above proposition, we see that the component of local stable trace formula and the component of global stable formula are compatible. So we will study the stable local trace formula in the Archimedean case. Therefore, there are two ways to study the stable distribution $S_M^G(\delta, f_\infty)$. One way is to explicitly stabilize the invariant trace formula in this special case. And the other is to stabilize the local trace formula, then we compare

the stable Weyl integral formula with the stable local trace formula. In this paper we will choose the second way.

5. THE TRANSFER FACTORS OF SPECTRAL SIDE AND CHARACTERS

Suppose that G is a reductive K -group over \mathbb{R} , Z stands for a central induced torus in G over \mathbb{R} , ζ is a character on $Z(\mathbb{R})$. Before we discuss the stabilization, we need to recall the invariant local trace formula [4] and virtual characters in the Archimedean case [5]. We set $V = \{\infty_1, \infty_2\}$ for two Archimedean places. Then $G_V = G(F_V) = G(\mathbb{R}) \times G(\mathbb{R})$ and $\zeta_V = \zeta \times \zeta^{-1}$, while $f = f_1 \times \bar{f}_2$, where $f_1, f_2 \in C(G(\mathbb{R}), \zeta)$. Then f is a function in the Schwartz space $C(G_V, \zeta_V)$.

The geometric side of the local trace formula is the linear form

$$(5.1) \quad I(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G-\text{reg}, \text{ell}}(M, V, \zeta)} I_M(\gamma, f) d\gamma$$

where $\Gamma_{G-\text{reg}, \text{ell}}(M, V, \zeta) = \{(\gamma, \gamma) : \gamma \in \Gamma_{G-\text{reg}, \text{ell}}(M, \zeta)\}$. (The set $\Gamma_{G-\text{reg}, \text{ell}}(M, V, \zeta)$ is in bijection with the family $\Gamma_{G-\text{reg}, \text{ell}}(\bar{M})$ of strongly G -regular, elliptic conjugacy classes in $\bar{M}(\mathbb{R}) = M(\mathbb{R})/Z(\mathbb{R})$).

The spectral side is the linear form

$$(5.2) \quad I(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{T_{\text{disc}}(M, V, \zeta)} i^M(\tau) I_M(\tau, f_1 \times \bar{f}_2) d\tau$$

where $I_M(\tau, f_1 \times \bar{f}_2) = r_M(\tau, P) \theta(\tau, f_{1,P}) \overline{\theta(\tau, f_{2,P})}$ as in [5], $T_{\text{disc}}(M, V, \zeta)$ stands for the diagonal image $\{(\tau, \tau^\vee) : \tau \in T_{\text{disc}}(M, \zeta)\}$ in $T_{\text{temp}}(M_V, \zeta_V)$. Sometimes we also denote $T_{\text{disc}}(M, V, \zeta)$ as $T_{\text{disc}}(M, \zeta)$ which depends on how we write the distribution. The subset $T_{\text{disc}}(M, \zeta)$ of $T_{\text{temp}}(M, \zeta) = T_{\text{temp}}(M(\mathbb{R}), \zeta)$ is defined in [5, §3].

We denote the leading term (the meaning is that $M=G$) of (5.2) by $I_{\text{disc}}(f)$. Then

$$(5.3) \quad I_{\text{disc}}(f) = \int_{T_{\text{disc}}(G, V, \zeta)} i^G(\tau) f_G(\tau) d\tau$$

where $f_G(\tau) = (f_1)_G(\tau) (\bar{f}_2)_G(\tau^\vee) = f_{1,G}(\tau) \overline{f_{2,G}(\tau)}$, and

$$i^G(\tau) = |W_\pi^0|^{-1} |R_{\pi,r}|^{-1} \sum_{w \in W_\pi(r)_{\text{reg}}} \varepsilon_\pi(w) |\det(1-w)_{a_M^G}|^{-1}.$$

Here $\tau = (M, \pi, r)$, W_π^0 is a subgroup of elements $w \in W_\pi$ such that the operator $R(w, \pi)$ (see [5, §2]) is a scalar, $W_\pi = \{w \in W(\mathfrak{a}_M) : w\pi \cong \pi\}$, $R_\pi = W_\pi/W_\pi^0$, $W_\pi(r)_{\text{reg}}$ is the intersection of the W_π^0 -coset $W_\pi(r) = W_\pi^0 r$ in W_π with the set $W_{\pi, \text{reg}} = \{w \in W_\pi : a_M^w = a_G\}$ of regular elements, $\varepsilon_\pi(w)$ stands for the sign of projection of w onto the Weyl group W_π^0 , taken relative to the decomposition $W_\pi = W_\pi^0 \times R_\pi$, $R_{\pi,r}$ is the centralizer of r in the group R_π .

If $f = f_1 \times \bar{f}_2$ and $f_1 \in C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, $f_2 \in C(G(\mathbb{R}), \zeta)$, then

$$I(f) = I_{\text{disc}}(f),$$

and $i^G(\tau) = |d(\tau)|^{-1} |R_{\pi,r}|^{-1}$, $d(\tau) = d(r) = \det(1-r)_{a_M/a_G}$.

We write \widehat{G} for the complex dual of G , and ${}^L G$ for the L-group $\widehat{G} \rtimes W_{\mathbb{R}}$, which acts through $W_{\mathbb{R}} \rightarrow \Gamma$, where the Weil group $W_{\mathbb{R}} = \mathbb{C}^\times \rtimes \Gamma$, Galois group $\Gamma = \{1, \sigma\}$.

An endoscopic data for G is a tuple $(G', \mathcal{G}', s', \xi)$, where

- (1) G' is a K -group and quasi-split over \mathbb{R} , and so has dual Galois automorphism $\sigma_{\widehat{G}'}$.
- (2) \mathcal{G}' is a split extension of $W_{\mathbb{R}}$ by \widehat{G}' , where $W_{\mathbb{R}}$ acts through $W_{\mathbb{R}} \rightarrow \Gamma$, and σ act as $\sigma_{\widehat{G}'}$ only up to an inner automorphism of \widehat{G}' .
- (3) s' is a semisimple element of \widehat{G} ,
- (4) $\xi' : \mathcal{G}' \rightarrow {}^L G$ is an embedding of extensions under which the image of \widehat{G}' is the identity component of $\text{Cent}(s', \widehat{G})$, and the full image lies in $\text{Cent}(s'', {}^L G)$, for some s'' congruent to s' modulo the center $Z(\widehat{G})$ of \widehat{G} .

We denote $\mathcal{E}(G)$ for the set of equivalence of endoscopic data for G . Then $\mathcal{E}(G) = \coprod_{\{M\}} (\mathcal{E}_{\text{ell}}(M)/W(M))$ or $\mathcal{E}(G) = (\coprod_{G' \in \mathcal{E}_{\text{ell}}} \mathcal{L}^{G'}) / \sim$, where the equivalence relation is defined by \widehat{G} conjugacy.

We first recall the geometric side of stable local trace formula, which is given by Arthur [11, § 6]. We shall identify G' with the diagonal endoscopic datum $G'_V = G' \times \bar{G}'$ for $G_V = G \times G$, where G' represents the datum $(G', \mathcal{G}', s', \xi')$, and \bar{G}' represents the adjoint datum $(G', \mathcal{G}', (s')^{-1}, \xi')$. The Langlands-Shelstad transfer factor attached to (G, G') depends on a choice of auxiliary data $\widehat{G}' \rightarrow G'$ and $\tilde{\xi}' : \mathcal{G}' \rightarrow {}^L \bar{G}'$ for G' . We can choose a compatible auxiliary data for \bar{G}' , so that the relative transfer factor for (G, \bar{G}') is the inverse of the relative transfer factor for (G, G') , then we have transfer property, $\bar{f}_2^{\bar{G}'} = \overline{f_2^{G'}}$. The transfer mappings were used [8] to construct supplementary linear forms $I^\mathcal{E}(f)$ and $S^G(f)$ from $I(f)$, where

$$I^\mathcal{E}(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}^0(G)} \iota(G, G') \widehat{S}'(f') + \varepsilon(G) S^G(f),$$

in which the linear forms $\widehat{S}' = \widehat{S}^{\bar{G}'}$ on $SIC(\widetilde{G}'_V, \widetilde{\zeta}'_V)$ are determined inductively by the further requirement that $I^\mathcal{E}(f) = I(f)$ if G is a quasi-split group, where $\iota(G, G') = |\text{Out}_G(G')|^{-1} |Z(\widehat{G}')^\Gamma / Z(\widehat{G})^\Gamma|^{-1}$, and

$$\varepsilon(G) = \begin{cases} 1 & \text{if } G \text{ is quasi-split,} \\ 0 & \text{otherwise.} \end{cases}$$

If G is a general group, we have $I(f) = I^\mathcal{E}(f)$ [11, §6]. If G is quasi-split, we have a geometric expansion [11, (10.11)]

$$S^G(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Delta_{G-\text{reg}, \text{ell}(\widetilde{M}', V, \zeta')}} n(\delta)^{-1} S_M^G(\delta, f) d\delta,$$

If $f = f_1 \times \bar{f}_2$, $f_1 \in C_{\text{cusp}}(G, \zeta)$, $f_2 \in C(G, \zeta)$, then

$$S_M^G(\delta, f_1 \times \bar{f}_2) = S_M^G(\delta, f_1) \times \overline{f_2^G(\delta)}.$$

The spectral side of the local trace formula was stabilized in [8], when the test function $f \in C_{\text{cusp}}(G, \zeta) \times C(G, \zeta^{-1})$. But it is just a formal formula that matches the geometric side. The point of the present work is to directly construct the spectral side of the stable local trace formula. We need to review the fact of the spectral side of invariant trace formula, before we are going to make a stabilization.

The irreducible tempered characters could well be regarded as the objects dual to semisimple conjugacy classes in $G(\mathbb{R})$. It is better to take the family of virtual characters studied in [5], this family is parametrized by a set $\tilde{T}(G, \zeta)$.

Definition 5.1. $\tilde{T}(G, \zeta)$ is the set of W_0 orbits of (essential) triplets $\tau = (M, \pi, r)$, $M \in \mathcal{L}$, $\pi \in \Pi_2(M, \zeta)$, $r \in \widetilde{R_\pi}$, where $\Pi_2(M, \zeta)$ stands for the equivalence classes of irreducible unitary representations of $M(\mathbb{R})$ which are square integrable modulo the center, whose central character is ζ , and $\widetilde{R_\pi}$ is a fixed central extension

$$1 \longrightarrow \widetilde{Z_\pi} \longrightarrow \widetilde{R_\pi} \longrightarrow R_\pi \longrightarrow 1$$

of the R -group of π .

The purpose of the extension is to ensure that the normalized intertwining operators $r \mapsto \widetilde{R_P}(r, \pi)$, $r \in \widetilde{R_\pi}$, $P \in \mathcal{P}(M)$ for the induced representation $I_P(\pi)$ give a representation of $\widetilde{R_\pi}$ instead of just a projective representation of R_π . However, in the Archimedean case, R_π is a product of groups $\mathbb{Z}/2\mathbb{Z}$, the cocycle which defines $\widetilde{R_\pi}$ splits, so we take $\widetilde{R_\pi} = R_\pi$.

There is a bijection $\rho \mapsto \pi_\rho$ from $\Pi(R_\pi)$ the set of irreducible representations of R_π onto the set of irreducible constituents of $I_P(\pi)$, with the properties that

$$(5.4) \quad \Theta(\tau, f) = \text{tr}(R_P(r, \pi)I_P(\pi, f)) = \sum_{\rho \in \Pi(R_\pi)} \text{tr}(\rho^\vee(r)) \text{tr}(\pi_\rho(f)).$$

and

$$(5.5) \quad \text{tr}(\pi_\rho(f)) = |R_\pi|^{-1} \sum_{r \in R_\pi} \text{tr}(\rho(r)) \text{tr}(R_P(r, \pi)I_P(\pi, f)).$$

There is an action $\tau \mapsto w\tau = (wM, w\pi, wr)$, $w \in W_0^G$, of W_0^G on the set of all triplets, with the property that $\Theta(w\tau, f) = \Theta(\tau, f)$. We can write

$$T(G, \zeta) = \coprod_{\{M\}} (T_{\text{ell}}(M, \zeta)/W(M)),$$

where $T_{\text{ell}}(M, \zeta) = \{\tau \in T(G, \zeta) : a_M^r = a_G, r \in R_\pi\}$, $\{M\}$ as usual runs over the orbits in \mathcal{L}/W_0 . There is also an action $\tau \mapsto \tau_\lambda = (G, \pi_\lambda, r)$, $\tau \in T_{\text{ell}}(G, \zeta)$, $\lambda \in ia_{G, Z}^*$ of $ia_{G, Z}^*$ on $T_{\text{ell}}(G, \zeta)$, where $\pi_\lambda(x) = \pi(x)e^{\lambda(H_G(x))}$ for any $x \in G(F)$. There is a similar action of $ia_{M, Z}^*$ on $T_{\text{ell}}(M, \zeta)$ for each M . This gives $T(G, \zeta)$ the structure of a disjoint union of finite quotients of compact tori. We will write $T(G, \zeta)_{\mathbb{C}}$ to be the disjoint union over $\{M\}$ of the spaces of $W(M)$ orbits in $T_{\text{ell}}(M, \zeta)_{\mathbb{C}} = \{\tau_\lambda : \tau \in T_{\text{ell}}(M, \zeta), \lambda \in a_{M, Z, \mathbb{C}}^*\}$, where $a_{M, Z}^*$ is the subspace of linear forms on a_M that are trivial on the image of a_Z in a_M , and $a_M = X(M)_{\mathbb{R}} \otimes \mathbb{R}$, where $X(M)$ is the set of rational characters on the M .

We denote $T_{\text{disc}}(G, \zeta)$ as a set of orbits (M, π, r) in $T(G, \zeta)$ such that $W_\pi(r)_{\text{reg}}$ is not empty. Then

$$T_{\text{ell}}(G, \zeta) \subset T_{\text{disc}}(G, \zeta) \subset T(G, \zeta).$$

Let $I(G(\mathbb{R}), \zeta)$ be the space of functions

$$\alpha : T(G, \zeta) \rightarrow \mathbb{C},$$

which satisfy the following three conditions;

- (1) α is supported on finitely many components of $T(G, \zeta)$,
- (2) α is symmetric under W_0^G ,
- (3) $\alpha \in S(T(G, \zeta))$.

Here $S(T(G, \zeta))$ is the space of smooth functions α on $T(G, \zeta)$, such that for each $M \in \mathcal{L}$, each integer n and each invariant differential operator $D = D_\lambda$ on $ia_{M, Z}^*$ transferred in the obvious way $D_\tau \alpha(\tau) = \lim_{\lambda \rightarrow 0} D_\lambda \alpha(\tau_\lambda)$, $\tau \in T_{\text{ell}}(M, \zeta)$ to $T_{\text{ell}}(M, \zeta)$, the semi-norm

$$\|\alpha\|_{M, D, n} = \sup_{\tau \in T_{\text{ell}}(M, \zeta)} (|D_\tau \alpha(\tau)| (1 + \|\mu_\tau\|)^n)$$

is finite, where $\mu_\tau = \mu_\pi$ for $\tau = (M, \pi, r)$, μ_π is the linear form determines the infinitesimal character of π . Then there is a natural topology which makes $I(G(\mathbb{R}), \zeta)$ into a complete topological vector space. By means of the inversion formula (5.5), we can identify $I(G(\mathbb{R}), \zeta)$ with the topological vector space of functions on $\Pi_{\text{temp}}(G(\mathbb{R}), \zeta)$, and also denoted by $I(G(\mathbb{R}), \zeta)$.

The trace Paley-Wiener theorem [6] is equivalent to the assertion that the map which sends $f \in C(G(\mathbb{R}), \zeta)$ to the function $f_G(\tau) = \Theta(\tau, f)$ is an open, continuous and surjective linear transformation from $C(G(\mathbb{R}), \zeta)$ onto $I(G(\mathbb{R}), \zeta)$. Observe that if $\tau \in T_{\text{ell}}(G, \zeta)$, there is a function $f \in C(G(\mathbb{R}), \zeta)$ with $f_G(\tau) = 1$, and such that f_G vanishes away from the $ia_{G, Z}^*$ orbit of $\tau \in T(G, \zeta)$. We call such a function f for a pseudo-coefficient of τ .

Stabilization of spectral side of local trace formula depends on the Shelstad's recent work. She [28], [29] directly constructs the spectral transfer factors and obtains the transfer theorem, the adjoint relation on K -group and the structure of tempered L -packets are given in [30].

We denote by $\Pi_{\text{temp}}(G, \zeta)$ the set of tempered representations, with central character equal to ζ . We have $\Pi_{\text{temp}}(G, \zeta) = \coprod_{\{M\}} \Pi_2(M, \zeta)/W(M)$, here $\{M\}$ for the set of W_0^G -orbits of Levi subgroups of G .

The stable character on $G(\mathbb{R})$ is attached to Langlands parameters. $\phi : W_{\mathbb{R}} \rightarrow {}^L G$ is an L -homomorphism, which maps from $W_{\mathbb{R}} = \text{Gal}(\mathbb{C}/\mathbb{R}) \rtimes \mathbb{C}^\times$ into the L -group ${}^L G$. We denote $\Phi(G)$ for the set of \widehat{G} orbits of parameters which are tempered, which means that the image of $W_{\mathbb{R}}$ in \widehat{G} is bounded. We denote $\Phi_2(G)$ for the subset of parameters in $\Phi(G)$ which are cuspidal, which means that the image of $W_{\mathbb{R}}$ is contained in no proper parabolic subgroup. There is a canonical decomposition

$$\Phi(G) = \coprod_{\{M\}} (\Phi_2(M)/W(M)).$$

For any ϕ , we denote S_ϕ as the centralizer in \widehat{G} of the image of ϕ , and \mathcal{S}_ϕ stands for the group of connected components in $\bar{S}_\phi = S_\phi/Z(\widehat{G})^\Gamma$. We say ϕ is elliptic, if $\bar{S}_{\phi,s}$ is finite for some semisimple element $s \in \bar{S}_\phi$. Any parameter $\phi \in \Phi(G)$ has a central character ζ on $Z(\mathbb{R})$, whose Langlands parameter is just the composition

$$W_F \xrightarrow{\phi} {}^L G \rightarrow {}^L Z.$$

The entire set $\Phi(G)$ decomposes into a disjoint union over ζ of the subsets $\Phi(G, \zeta)$ of parameters with central character ζ . The set $\Phi_2(G, \zeta)$ also comes with an action $\phi \mapsto \phi_\lambda = \phi \circ \rho_\lambda$ of $ia_{G,Z}^*$, where $\rho_\lambda \in H^1(W_\mathbb{R}, Z(\widehat{G})^\Gamma)$, which corresponds to the character $\pi_{\rho_\lambda}(x) = e^{\lambda(H_G(x))}$.

Given $s \in \mathcal{S}_\phi$, we attach an endoscopic data $G' = G^s = (G^s, \mathcal{G}^s, s, \xi^s)$, where \mathcal{G}^s is the subgroup of ${}^L G$ generated by $\text{Cent}(s, \widehat{G})^\circ$ and the image of ϕ , ξ^s is the inclusion $\mathcal{G}^s \hookrightarrow {}^L G$ and G^s is a quasi-split group. Usually, \mathcal{G}^s need not be an L -group, that is there might not be an L -isomorphism from \mathcal{G}^s to ${}^L G^s$ which is the identity on \widehat{G}' . To deal with the problem, we need to make a z -extension \widetilde{G}' of G' . For simplicity, we assume $\widetilde{G}' = G'$, for any G' .

Shelstad established a spectral transfer mapping, which given by a linear combination

$$(5.6) \quad f'(\phi') = \sum_{\pi \in \Pi_{\text{temp}}(G(\mathbb{R}))} \Delta(\phi', \pi) f_G(\pi)$$

of irreducible tempered characters $f_G(\pi) = \text{tr}(\pi(f))$, $\pi \in \Pi_{\text{temp}}(G)$, on $G(\mathbb{R})$. The coefficients are spectral transfer factors $\Delta(\phi', \pi)$. They are established explicitly by Shelstad in [29], which are compatible with the geometric transfer factors. We assume implicitly that the Langlands parameter ϕ is relevant to G , in the sense that if its image is contained in a parabolic subgroup ${}^L P \subset {}^L G$, then ${}^L P$ is dual to a \mathbb{Q} -rational parabolic subgroup $P \subset G$. It then gives rise to the L -packet Π_ϕ that was an integral part of Langlands's classification of representations of real groups [22]. Π_ϕ is a finite subset of representations in $\Pi_{\text{temp}}(G)$ whose constituents have the same local L -functions and ε -factors, and that $\Pi_{\text{temp}}(G)$ is a disjoint union over ϕ of the subsets Π_ϕ . Shelstad observed that for any ϕ , the distribution

$$f^G(\phi) = \sum_{\pi \in \Pi_\phi} f_G(\pi)$$

is stable, in the sense that it depends only on the image f^G of f in $S(G)$. $S(G) = \{f^G : f \in C(G)\}$, where f^G is that stable orbital integral.

$$f^G(\delta) = |D(\delta)|^{1/2} \int_{G_\delta(\mathbb{R}) \backslash G(\mathbb{R})} f(x^{-1}\delta x) dx = \sum_{\gamma \rightarrow \delta} f_G(\gamma)$$

Applied to G' instead of G , this gives meaning to the left hand side of (5.6). We also assume that the given pair (G', ϕ') is relevant to G , in the sense that the composite Langlands parameter $\phi = \xi' \circ \phi' : W_\mathbb{R} \rightarrow {}^L G$ is relevant to G . Then we have a bijection

mapping

$$(5.7) \quad (G', \phi') \rightarrow (\phi, s),$$

where $s \in \mathcal{S}_\phi$.

Shelstad also established an inversion of transfer mapping, which is given by a linear combination

$$f_G(\pi) = \sum_{s_{\text{sc}}} \Delta(\pi, \phi^s) f'(\phi^s)$$

of stable endoscopic characters, where $s_{\text{sc}} \in \widetilde{\mathcal{S}}_\phi$ comes from an extension

$$1 \rightarrow \widehat{Z}_{\text{sc}} \rightarrow \widetilde{\mathcal{S}}_\phi = \Pi_0(S_{\phi, \text{sc}}) \rightarrow \mathcal{S}_\phi \rightarrow 1,$$

where $S_{\phi, \text{sc}}$ is the preimage of \bar{S}_ϕ in \widehat{G}_{sc} , the simply connected cover the derived group of \widehat{G} and $\widehat{Z}_{\text{sc}} = Z(\widehat{G}_{\text{sc}})$, ϕ^s comes from the map (5.7). The inversion rests on explicit adjoint relations for spectral transfer factors Δ_{spec} defined initially as a product $\Delta_I \Delta_{II} \Delta_{III}$ in the G -regular case in [29], and we have adjoint relations

$$\sum_{s_{\text{sc}}} \Delta(\pi, \phi^s) \Delta(\phi^s, \pi') = \delta(\pi, \pi'),$$

where the summing over semisimple representative s_{sc} for $\widetilde{\mathcal{S}}_\phi / \text{Ker}(\widetilde{\mathcal{S}}_\phi \rightarrow \mathcal{S}_\phi) \simeq \mathcal{S}_\phi$, $\delta(.,.)$ denotes the Kronecker delta function, $\Delta(\pi, \phi^s) = \frac{1}{n(\pi)} \Delta(\phi^s, \pi)^{-1}$, and $\Delta(\phi^s, \pi)^{-1} = \overline{\Delta(\phi^s, \pi)} / \|\Delta\|^2$, $n(\pi) = |\mathcal{S}_\phi|$ is the cardinality of the L -packet of π , $\|\Delta\| = |\Delta(\phi^s, \pi)|$ is a constant (independent s), which is compatible with geometric transfer factor. However, geometric transfer factor is unitary, so we take $\|\Delta\| = 1$.

We also obtain the other adjoint relations

$$\sum_{\pi \in \Pi_\phi} \Delta(\phi^s, \pi) \Delta(\pi, \phi^{s'}) = \delta(s, s'),$$

where $\phi = \xi^s \circ \phi^s$, s, s' in \mathcal{S}_ϕ .

Naturally, we can define an inversion adjoint transfer factor

$$\Delta(\tau, \phi^s) = \sum_{\chi \in \widehat{R}_\pi} \overline{\chi(r)} \Delta(\pi^\chi, \phi^s),$$

where $\overline{\chi(r)} = \text{tr}(\rho^\vee(r))$. because R_π is finite abelian group.

We obtain

$$(5.8) \quad \Theta(\tau, f) = \sum_{s \in \mathcal{S}_\phi} \Delta(\tau, \phi^s) f'(\phi^s).$$

However, when we define the transfer factor $\Delta(\phi^s, \tau)$, we need to assume the Langlands parameter ϕ to be elliptic. This means that Π_ϕ contains elliptic representations. ϕ factors through a discrete parameter for a cuspidal Levi subgroup ${}^L M$, and so through ${}^L T_M$ [22],

where T_M is the maximal torus which is compact modulo the center of M . We consider the associated short exact sequence [29]

$$1 \rightarrow \mathcal{E}(T_M) \rightarrow \mathcal{S}_\phi \rightarrow R_\phi \rightarrow 1,$$

where R_ϕ is Langlands R -group, and the group $\mathcal{E}(T_M)$ is isomorphic to \mathcal{S}_{ϕ_M} , where M is a Levi subgroup of G , and $\phi_M : W_{\mathbb{R}} \rightarrow {}^L M$ is a Langlands parameter for M whose image in ${}^L G$ equals ϕ , and whose L -packet Π_{ϕ_M} consists of representations in the discrete series of $M(\mathbb{R})$. Suppose $\pi_M \in \Pi_{\phi_M}$ corresponds to the character χ on the group \mathcal{S}_{ϕ_M} . Since \mathcal{S}_ϕ is abelian, R_χ equals the full group R_ϕ , where R_χ is the subgroup of elements in R_ϕ that stabilize χ , and χ extends to a character θ on \mathcal{S}_ϕ . The set of such θ is a torsor under the action of the characters in R_ϕ . It corresponds to subset Π_{ϕ, π_M} of Π_ϕ , composed of the irreducible constituents of the induced representation $I_P^G(\pi_M)$, where P belongs to the set $\mathcal{P}(M)$ of parabolic subgroups of G with Levi component M , so we have $|R_{\pi_M}| = |R_\phi|$. On the other hand, we identify the stabilizer W_ϕ of ϕ_M with a subgroup of $W(M)$, the Weyl group that contains the stabilizer W_{π_M} of π_M . It is a consequence of the disjointness of tempered L -packets for M that W_ϕ contains W_{π_M} . From the above discussion, we know that any element in W_ϕ stabilizes π_M . Therefore W_ϕ equals W_{π_M} . Moreover, we know that elements in the subgroup W_ϕ^0 of W_ϕ give scalar intertwining operators for the induced representation $I_P^G(\pi_M)$. It follows that W_ϕ^0 is contained in the subgroup $W_{\pi_M}^0$ of W_{π_M} . We have a surjective mapping

$$R_\phi \cong W_\phi / W_\phi^0 \mapsto W_{\pi_M} / W_{\pi_M}^0 \cong R_{\pi_M}, \quad \pi_M \in \Pi_{\phi_M},$$

which make sense for any parameter $\phi \in \Phi_{\text{ell}}(G, \xi)$. So we obtain $R_{\pi_M} = R_\phi$. We define a subset $T_\phi = \{(M, \pi, r) : M = M_\phi, \pi \in \Pi_{\phi_M}, r \in R_\phi\}$ of $T(G)$, then $|T_\phi| = |\Pi_{\phi_M}| |R_\phi| = |\mathcal{S}_\phi| = |\Pi_\phi|$. And we get a bijection from T_ϕ to \mathcal{S}_ϕ .

We can define the adjoint transfer factor

$$\Delta(\phi^s, \tau) = \sum_{\chi \in \widehat{R}_\pi} \frac{1}{|R_\pi|} \chi(r) \Delta(\phi^s, \pi^\chi),$$

where $\tau = (M, \pi, r)$, R_π is a 2-group, and π^χ is the irreducible component of induce representations of π , which corresponds to χ by Arthur's classification Theorem in [5, §2]. So we obtain

$$(5.9) \quad f'(\phi^s) = \sum_{\tau \in T_\phi} \Delta(\phi^s, \tau) \Theta(\tau, f),$$

The transfer factors have the following properties.

Proposition 5.2. *If ϕ is elliptic, then*

(1)

$$\Delta(\tau, \phi^s) = \frac{|R_\phi|}{|\mathcal{S}_\phi|} \overline{\Delta(\phi^s, \tau)},$$

(2) we have the adjoint relations,

$$(5.10) \quad \sum_{\tau \in T_\phi} \Delta(\phi^{s_1}, \tau) \Delta(\tau, \phi^{s_2}) = \delta(\phi^{s_1}, \phi^{s_2}),$$

$$(5.11) \quad \sum_{s \in \mathcal{S}_\phi} \Delta(\tau, \phi^s) \Delta(\phi^s, \tau_1) = \delta(\tau, \tau_1).$$

Proof. We first check (1). It is easy from the definition.

$$\Delta(\tau, \phi^s) = \sum_{\chi \in \widehat{R}_\pi} \overline{\chi(r)} \Delta(\pi^\chi, \phi^s),$$

and

$$\overline{\Delta(\phi^s, \tau)} = \sum_{\chi \in \widehat{R}_\pi} \frac{1}{|R_\pi|} \overline{\chi(r) \Delta(\phi^s, \pi^\chi)} = \sum_{\chi \in \widehat{R}_\pi} \frac{n(\pi)}{|R_\pi|} \overline{\chi(r)} \Delta(\pi^\chi, \phi^s).$$

Where $|R_\pi| = |R_\phi|$, $n(\pi) = |\mathcal{S}_\phi|$, we obtain the equation (1).

We now prove (2). We only check (5.10), as (5.11) is similar. Observe that ϕ is relevant to G , and

$$f'(\phi^{s_1}) = \sum_{\tau \in T_\phi} \Delta(\phi^{s_1}, \tau) \Theta(\tau, f) = \sum_{s \in \mathcal{S}_\phi} \sum_{\tau \in T_\phi} \Delta(\phi^{s_1}, \tau) \Delta(\tau, \phi^s) f'(\phi^s).$$

Using the fact that the characters of representations are linear independent, then we obtain the identity (5.8). \square

We return to analyze the transfer factors

$$\Delta(\phi^s, \tau) = \sum_{\chi \in \widehat{R}_\pi} \frac{1}{|R_\pi|} \chi(r) \Delta(\phi^s, \pi^\chi),$$

where $\chi(r)$ is simple enough, it is the character of finite abelian group. $\Delta(\phi^s, \pi^\chi)$ is defined directly by Shelstad in [29], and extended to K -group [30], She also checked Arthur's conjecture about the transfer factors in [12], and obtain a nice formula about $\Delta(\phi^s, \pi^\chi)$ in [30]. $\Delta(\phi^s, \pi^\chi) = \rho(\Delta, s_{\text{sc}}) \langle s_{\text{sc}}, \pi^\chi \rangle$, where $\rho(\Delta, s_{\text{sc}})$ satisfies $\rho(t\Delta, z_{\text{sc}} s_{\text{sc}}) = t \rho(\Delta, s_{\text{sc}}) \zeta_G(z_{\text{sc}})^{-1}$ for $t \in \mathbb{C}^\times$, and $z_{\text{sc}} \in Z(\widehat{G}_{\text{sc}})$, s_{sc} is the image s of the mapping $\widehat{\mathcal{S}}_\phi \rightarrow \mathcal{S}_\phi$. ζ_G comes from Arthur's paper [8]. And $\rho(\Delta, s_{\text{sc}}) = \zeta(s_{\text{sc}})^{-1} \delta(\pi^s, \pi^{\text{base}})$, $\zeta(s_{\text{sc}}) = \langle s_{\text{sc}}, \pi^{\text{base}} \rangle$.

If G is quasi-split, $\rho(\Delta, s_{\text{sc}}) = 1$, the transfer factors $\Delta(\phi^s, \pi^\chi) = \langle s_{\text{sc}}, \pi^\chi \rangle$ is simple enough. In section 6, I will stabilize the spectral side of the invariant local trace formula when the component of test function is cuspidal, which is enough to give the multiplicity formula. In section 7, we will obtain the stable Weyl integral formula and the stable distributions.

6. STABILIZATION OF THE ELLIPTIC TERM

In the present section, we shall consider the test function $f = f_1 \times \bar{f}_2$, $f_1 \in C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, $f_2 \in C(G(\mathbb{R}), \zeta)$, G is K -group over \mathbb{R} . Recall that $C_{\text{cusp}}(G(\mathbb{R}), \zeta)$ stands for the space of functions f_1 in $C(G(\mathbb{R}), \zeta)$ that are cuspidal, in the sense that the orbital integral

$$\gamma \mapsto f_{1,G}(\gamma) = J_G(\gamma, f_1), \quad \gamma \in \Gamma(G)$$

is supported on the subset $\Gamma_{\text{ell}}(G)$ of elliptic classes in $\Gamma(G)$.

We assume that f_1 is cuspidal. We set

$$I_{\text{disc}}(f) = \int_{T_{\text{ell}}(G, \zeta)} i^G(\tau) f_{1,G}(\tau) \overline{f_{2,G}(\tau)} d\tau,$$

where $i^G(\tau) = |R_{\pi,r}|^{-1} |\det(1-r)_{a_M/a_G}|^{-1}$, $\tau = (M, \pi, r)$, $T_{\text{ell}}(G, \zeta) = \prod_{\alpha \in \pi_0(G)} T_{\text{ell}}(G_\alpha, \zeta_\alpha)$. For the given f , $I_{\text{disc}}(f)$ equals the spectral side of the local trace formula. We have

$$I(f) = I_{\text{disc}}(f).$$

Now we can regard I_{disc} as a linear form on the subspace

$$C_{1-\text{cusp}}(G_V, \zeta_V) = C_{\text{cusp}}(G, \zeta) \otimes C(G, \zeta^{-1})$$

of $C(G_V, \zeta_V)$.

If $f_1 \in C_{\text{cusp}}(G, \zeta)$, then f_1 is supported on $T_{\text{ell}}(G, \zeta)$. For stabilization, we need to consider Langlands parameters ϕ which are elliptic. We take the test function f for cuspidal on the equation (5.8) and (5.9), by the Shelstad transfer theorem [27], we can define a corresponding set

$$\Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta) = \{(G', \phi') : G' \in \mathcal{E}_{\text{ell}}(G), \phi' \in \Phi_2(G', G, \zeta)\},$$

where $\Phi_2(G', G, \zeta) = \Phi_2(G, \zeta) / \text{Out}_G(G')$, $\text{Out}_G(G') = \text{Aut}_G(G') / \zeta'(\widehat{G'})$, $\text{Aut}_G(G') = \{g \in \widehat{G} : gs'g^{-1} \in s'Z(\widehat{G}), g\mathcal{G}'g^{-1} = \mathcal{G}'\}$. We denote $SI_{\text{cusp}}(G', \zeta)$ for the set of linear forms $f'(\phi')$ on $\Phi_2(G', \zeta)$ obtained from the transfer map. $f'(\phi')$ depends only on the image of ϕ' in $\Phi_2(G', G, \zeta)$, the set of $\text{Out}_G(G')$ -orbit in $\Phi_2(G', \zeta)$ [30]. If $f'(\phi') \in SI_{\text{cusp}}(G', \zeta)$, then $f'(\phi')$ is supported on $\Phi_2(G', G, \zeta)$.

We have

$$\begin{aligned} \theta(\tau, f) &= \sum_{\phi' \in \Phi_{\text{ell}}^{\mathcal{E}}(G)} \Delta(\tau, \phi') f^{\mathcal{E}}(\phi') \\ &= \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \sum_{\phi' \in \Phi_2(G', G, \zeta)} \Delta(\tau, \phi') f^{G'}(\phi'), \end{aligned}$$

and

$$f^{G'}(\phi') = \sum_{\tau \in T_{\text{ell}}(G, \zeta)} \Delta(\phi', \tau) \theta(\tau, f).$$

Lemma 6.1. *Both $\Delta(\tau, \phi)$ and $\Delta(\phi, \tau)$ have finite support in ϕ for fixed τ , and finite support in τ for fixed ϕ . Moreover,*

$$\sum_{\phi' \in \Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta)} \Delta(\tau, \phi') \Delta(\phi', \tau_1) = \delta(\tau, \tau_1), \quad \tau, \tau_1 \in T_{\text{ell}}(G, \zeta),$$

$$\text{and } \sum_{\tau \in T_{\text{ell}}(G, \zeta)} \Delta(\phi', \tau) \Delta(\tau, \phi'_1) = \delta(\phi', \phi'_1), \quad \phi', \phi'_1 \in \Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta).$$

where $\delta(\tau, \tau_1)$ and $\delta(\phi', \phi'_1)$ are Kronecker delta functions.

Proof. $\Delta(\tau, \phi)$ and $\Delta(\phi, \tau)$ have finite support, which is equivalent to say Shelstad's transfer factor $\Delta(\pi, \phi)$ and $\Delta(\phi, \pi)$ have finite support. This is obvious in [29, 7]. The proof of second part is similar to Proposition 5.2 (2). \square

We define a measure on $\Phi_2(G, \zeta)$ by setting

$$\int_{\Phi_2(G, \zeta)} \beta(\phi) d\phi = \sum_{\phi \in \Phi_2(G)/ia_{G,Z}^*} \int_{ia_{G,Z}^*} \beta(\phi_\lambda) d\lambda$$

for any $\beta \in C(\Phi_2(G, \zeta))$. So $\Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta)$ also has a measure from the quotient measures on the set $\Phi_2(G', G, \zeta)$ and the transfer factors can govern a change of variables of integration.

Lemma 6.2. *Suppose that $\alpha \in C(T_{\text{ell}}(G), \zeta)$, and that $\beta \in C_{\text{cusp}}(\Phi_{\text{ell}}^{\mathcal{E}}(G), \zeta)$. Then*

$$\begin{aligned} & \int_{T_{\text{ell}}(G, \zeta)} \sum_{\phi \in \Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta)} \beta(\phi) \Delta(\phi, \tau) \alpha(\tau) d\tau \\ &= \int_{\Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta)} \sum_{\tau \in T_{\text{ell}}(G, \zeta)} \beta(\phi) \Delta(\phi, \tau) \alpha(\tau) d\phi. \end{aligned}$$

Proof. According to the definition of the measure $d\tau$, we can decompose the left hand side of the required identity into an expression

$$\sum_{\tau} \int_{ia_{G,Z}^*} \sum_{\phi} \sum_{\mu} \beta(\phi_\mu) \Delta(\phi_\mu, \tau_\lambda) \alpha(\tau_\lambda) d\lambda.$$

Here $\tau \in T_{\text{ell}}(G, \zeta)/ia_{G,Z}^*$, $\phi \in \Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta)/ia_{G,Z}^*$, $\mu \in ia_{G,Z}^*$. We recall that the transfer factor $\Delta(\phi, \tau)$ vanishes unless $\tau \in T_\phi$. We observe from the definition and Langlands classification, $\phi_\mu = \phi \circ \rho_\mu$. Here $\rho_\mu \in H^1(W_k, Z(\widehat{G}))$ and it corresponds to the L -packet which is $\Pi_{\phi_\mu} = \{\pi_{\rho_\mu} \otimes \pi | \pi \in \Pi_\phi\}$, where $\pi_{\rho_\mu}(x) = e^{\mu(H_G(x))}$ for a central character. So μ determines a L -packet in [28]. However, $\tau_\lambda = (M, \pi_\lambda, r)$, where $\pi_\lambda(x) = \pi(x)e^{\lambda(H_G(x))}$. If $\Delta(\phi_\mu, \tau_\lambda)$ doesn't vanish, then ϕ_μ and τ_λ have the same parameter ϕ_λ . So, $\mu = \lambda$. We see that the sum over μ reduces to the one element $\mu = \lambda$. The expression becomes

$$\sum_{(\tau, \phi)} \int_{ia_{G,Z}^*} \beta(\phi_\lambda) \Delta(\phi_\lambda, \tau_\lambda) \alpha(\tau_\lambda) d\lambda.$$

where (τ, ϕ) is summed over pairs in $(T_{\text{ell}}(G, \zeta) \times \Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta))/ia_{G, Z}^*$. From its obvious symmetry, we conclude that the expression must also be equal to the right hand side of the required identity. The identity is therefore valid. \square

In the following, we will stabilize the elliptic term of spectrum side of invariant local trace formula and obtain the explicit coefficient.

Assume that f_1 is a cuspidal function, we denote $|d(\tau)| = |d(r)| = |\det(1 - r)_{a_M/a_G}|^{-1}$, $i^G(\tau) = |R_{\pi, r}|^{-1}|d(\tau)|^{-1}$. We need to stabilize $d(\tau)f_{1, G}(\tau)$.

Lemma 6.3. *If $f_1 \in C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, which is supported on $T_{\text{ell}}(G, \zeta)$. There exists a cuspidal function \tilde{f}_1 , satisfying*

$$\tilde{f}_{1, G}(\tau) = d(\tau)f_{1, G}(\tau).$$

Proof. Notice that $\tau \rightarrow d(\tau)f_{1, G}(\tau)$ is also a function in $I(G(\mathbb{R}), \zeta)$, which is supported on $T_{\text{ell}}(G, \zeta)$. The trace Paley-Wiener theorem therefore provides us with a cuspidal function $\tilde{f}_1 \in C(G(\mathbb{R}), \zeta)$ such that

$$\tilde{f}_{1, G}(\tau) = \theta(\tau, \tilde{f}_1) = d(\tau)f_{1, G}(\tau).$$

\square

For computing the coefficient of the spectral side of the stable local trace formula, we need to consider the two cases, one is that π is elliptic as well as regular, the other is that π is elliptic and not regular. We consider the first case, then π is a discrete series representation. Thus ϕ is discrete, which satisfies $\pi \in \Pi_{\phi}$, and the image of ϕ is contained in no proper parabolic subgroup of ${}^L G$. For each s_{sc} in $\mathcal{S}_{\phi}^{\text{sc}}$, the parameter ϕ must be G -regular and discrete, so that π^s is also a discrete series representations.

We need to recall the detail of Langlands parameter ϕ . We fixed a splitting $\text{spl}_{\hat{G}} = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ of \hat{G} , where \mathcal{B} is a borel subgroup in \hat{G} , \mathcal{T} is a maximal torus in \hat{G} included in \mathcal{B} , X_{α^\vee} is a eigenvector of α^\vee . Let ι be half of the sum of the roots of \mathcal{T} in \mathcal{B} . Then according to [22] or [27, 7], there is a pair $\mu, \lambda \in X_*(\mathcal{T}) \otimes \mathbb{C}$, which parameterize the Langlands parameter ϕ , and $\phi = \phi(\mu, \lambda)$ is defined by

$$\phi(z \times 1) = z^\mu \bar{z}^{\sigma_T(\mu)}$$

for $z \in \mathbb{C}^\times$, and

$$\phi(1 \times \sigma) = e^{2\pi i \lambda} n(\sigma_T) \times (1 \times \sigma),$$

with $n(\sigma_T)$ the element of \hat{G} attached by X_{α^\vee} to the Weyl group element $\omega(\sigma_T)$, where σ_T acts as $\omega(\sigma_T) \circ \sigma$ on \mathcal{T} , as in [24, section 2.1], and μ, λ satisfy the property

$$\frac{1}{2}(\mu - \sigma_T \mu) - \iota + (\lambda + \sigma_T \lambda) \in X_*(\mathcal{T}).$$

Here μ is determined uniquely, while λ is determined only modulo

$$X_*(\mathcal{T}) + \{\nu - \sigma_T \nu : \nu \in X_*(\mathcal{T}) \otimes \mathbb{C}\}$$

and ϕ is determined uniquely up to \mathcal{T} -conjugacy.

For endoscopic group $G' = (G', s, \mathcal{G}', \xi)$, where $\xi : \mathcal{G}' \rightarrow {}^L G$ is an embedding, and ξ can be parameterized by (μ^*, λ^*) . We assume that ${}^L G'$ equals \mathcal{G}' , $\phi' : W_{\mathbb{R}} \rightarrow {}^L G'$ is a Langlands parameter, and $\phi' = \phi'(\mu', \lambda')$, we have the commutative diagram $\phi = \xi \circ \phi'$. We have the relation of the Langlands parameter: $\mu = \mu' + \mu^*$, and $\lambda = \lambda' + \lambda^*$. If μ is dominant and regular, then π is a discrete series, which belongs to Π_{ϕ} , and $\pi' \in \Pi_{\phi'}$ is also a discrete series. In this case, we know the cardinal number of the L -packet Π_{ϕ} , which equals $|\mathcal{S}_{\phi}|$ in [30, Corollary 7.6], and

$$|\mathcal{S}_{\phi}| = |\mathcal{E}(T)|.$$

This case the Langlands R group R_{ϕ} is trivial. Then we obtain

$$\frac{|\mathcal{S}_{\phi}|}{|\mathcal{S}_{\phi'}|} = \frac{|\mathcal{E}(T)|}{|\mathcal{E}(T')|} = \frac{|\mathcal{K}_T|}{|\mathcal{K}_{T'}|} = \frac{|Z(\widehat{G'})^{\Gamma}|}{|Z(\widehat{G})^{\Gamma}|}.$$

Here $\mathcal{K}_T = \pi_0((\widehat{T})^{\Gamma}/Z(\widehat{G}^{\Gamma}))$, since the tori T and T' are isomorphic, then the third equality is holds, and the second equality comes from the Tate-Nakayama duality.

The following we deal with the singular elliptic case. In other words, when $\pi \in \Pi_{\phi}$ is elliptic representation, a Langlands parameter ϕ [22] factors through a discrete parameter for a cuspidal Levi subgroup ${}^L M$ and so through ${}^L T_M$, where the maximal torus T_M is compact modulo the center of M . Following the argument of [27], we construct a new Langlands parameter ϕ factoring through ${}^L T$, where T is compact modulo the center of G . This new Langlands parameter ϕ will be of the form $\phi(\mu, \lambda)$ as in the first case, but now the regularity requirement on μ is not necessary, we call such a ϕ for a limit of discrete parameters. We obtained, by transfer of a discrete parameter ϕ' for an elliptic endoscopy group, a limit of discrete parameters ϕ [18, section 14], so we obtain

$$\frac{|\mathcal{S}_{\phi}|}{|\mathcal{S}_{\phi'}|} = \frac{|Z(\widehat{G'})^{\Gamma}| |\text{Out}_G(G', \phi')|}{|Z(\widehat{G})^{\Gamma}|} = \frac{|Z(\widehat{G'})^{\Gamma}|}{|Z(\widehat{G})^{\Gamma}|}.$$

The second equality comes from the discrete case, where $\text{Out}_G(G', \phi')$ is the stabilizer of ϕ' (as a $\widehat{G'}$ -orbit) in the finite group $\text{Out}_G(G')$.

Lemma 6.4. *If ϕ is elliptic, then we have the coefficient relations*

$$\frac{|S_{\phi}|}{|S_{\phi^s}|} = |Z(\widehat{G'})^{\Gamma}/Z(\widehat{G})^{\Gamma}|$$

Theorem 6.5. *If $f = f_1 \times \bar{f}_2$, $f_1 \in C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, $f_2 \in C(G(\mathbb{R}), \zeta)$, then*

$$\begin{aligned} I_{\text{disc}}(f) &= \int_{T_{\text{ell}}(G, \zeta)} i^G(\tau) f_{1,G}(\tau) \overline{f_{2,G}(\tau)} d\tau \\ &= \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \widehat{S}^{G'}(f'), \end{aligned}$$

and $\widehat{S}^{G'}(f')$ is stable distribution, where

$$\begin{aligned} i^G(\tau) &= |d(\tau)|^{-1} |R_{\pi,r}|^{-1}, \\ \widehat{S}^{G'}(f') &= \int_{\Phi_2(G', \zeta)} S^{G'}(\phi') \widetilde{f}'_1(\phi') \overline{f'_2(\phi')} d\phi', \\ S^{G'}(\phi') &= \frac{1}{|\mathcal{S}_{\phi'}|}, \quad \phi = \xi' \circ \phi'. \end{aligned}$$

Proof. By (3.7) we have

$$(6.1) \quad I_{\text{disc}}(f) = \int_{T_{\text{ell}}(G, \zeta)} |R_{\pi,r}|^{-1} |d(\tau)|^{-1} f_{1,G}(\tau) \overline{f_{2,G}(\tau)} d\tau$$

Applying Lemma 6.3, then (6.1) equals to

$$\begin{aligned} & \int_{T_{\text{ell}}(G, \zeta)} |R_{\pi,r}|^{-1} \widetilde{f}_{1,G}(\tau) \overline{f_{2,G}(\tau)} d\tau \\ &= \int_{T_{\text{ell}}(G, \zeta)} |R_{\phi}|^{-1} \sum_{\phi \in \Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta)} \Delta(\tau, \phi) \widetilde{f}_1^{\mathcal{E}}(\phi) \overline{f_{2,G}(\tau)} d\tau \\ &= \int_{T_{\text{ell}}(G, \zeta)} |R_{\phi}|^{-1} \sum_{\phi \in \Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta)} \frac{|R_{\phi}|}{|\mathcal{S}_{\phi}|} \widetilde{f}_1^{\mathcal{E}}(\phi) \overline{\Delta(\phi, \tau) f_{2,G}(\tau)} d\tau \end{aligned}$$

Applying Lemma 6.2, We see that this last expression can be written as

$$\int_{\Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta)} |\mathcal{S}_{\phi}|^{-1} \widetilde{f}_1^{\mathcal{E}}(\phi) \sum_{\tau \in T_{\text{ell}}(G, \zeta)} \overline{\Delta(\phi, \tau) f_{2,G}(\tau)} d\phi,$$

which is just

$$\begin{aligned} & \int_{\Phi_{\text{ell}}^{\mathcal{E}}(G, \zeta)} |\mathcal{S}_{\phi}|^{-1} \widetilde{f}_1^{\mathcal{E}}(\phi) \overline{f_2^{\mathcal{E}}(\phi)} d\phi \\ &= \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \int_{\Phi_2(G', G, \zeta)} |\mathcal{S}_{\phi}|^{-1} \widetilde{f}'_1(\phi') \overline{f'_2(\phi')} d\phi'. \end{aligned}$$

We normalize the measure on the quotient $\Phi_2(G', G, \zeta)$ of $\Phi(G', \zeta)$ by the constant $|\text{Out}_G(G')|^{-1}$. We obtain

$$I_{\text{disc}}(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \int_{\Phi_2(G', G, \zeta)} |\mathcal{S}_{\phi}|^{-1} |Z(\widehat{G}')^{\Gamma} / Z(\widehat{G})^{\Gamma}| \widetilde{f}'_1(\phi') \overline{f'_2(\phi')} d\phi'$$

We denote the coefficient for $S^{G'}(\phi') = |\mathcal{S}_{\phi}|^{-1} |Z(\widehat{G}')^{\Gamma} / Z(\widehat{G})^{\Gamma}| = |\mathcal{S}_{\phi'}|^{-1}$, which comes from the Lemma 6.4.

Then we obtain the required formula

$$I_{\text{disc}}(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \widehat{S}^{G'}(f').$$

Since $\widetilde{f}_1'(\phi')$, $f_2'(\phi')$ are stable. So $\widehat{S}^{G'}(f')$ is stable. We have completely proved the theorem. \square

7. CHARACTERS AND STABLE ORBIT INTEGRAL

Assume that the test function $f = f_1 \times \bar{f}_2$, $f_1 \in C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, $f_2 \in (G(\mathbb{R}), \zeta)$, G is a reductive K -group. The key point in the stable trace formula is that the stable distribution $S_{M'}^{G'}(\delta, f)$ only depends on the quasisplit group G' which is independent of the group G . Then we have the following theorem.

Theorem 7.1. *If G is quasisplit group, we have*

$$S^G(f) = S_{\text{disc}}^G(f).$$

Here

$$(7.1) \quad S^G(f) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Delta_{G-\text{reg, ell}}(M, \zeta)} n(\delta)^{-1} S_M^G(\delta, f_1) \overline{f_2^G(\delta)} d\delta,$$

and

$$(7.2) \quad S_{\text{disc}}^G(f) = \int_{\Phi_2(G, \zeta)} S^G(\phi) \widetilde{f}_1^G(\phi) \overline{f_2^G(\phi)} d\phi.$$

Proof. Because we have

$$S^G(f) = I(f) - \sum_{G' \in \mathcal{E}_{\text{ell}}^\circ(G)} \iota(G, G') \widehat{S}^{G'}(f'),$$

and

$$S_{\text{disc}}^G(f) = I_{\text{disc}}(f) - \sum_{G' \in \mathcal{E}_{\text{ell}}^\circ(G)} \iota(G, G') \widehat{S}_{\text{disc}}^{G'}(f'),$$

and f_1 is cuspidal, then $I(f) = I_{\text{disc}}(f)$. We can prove this theorem inductively. G is quasisplit, then $\dim(G') < \dim(G)$ for all $G' \in \mathcal{E}_{\text{ell}}^\circ(G)$, so for all $G' \in \mathcal{E}_{\text{ell}}^\circ(G)$, we have $\widehat{S}^{G'}(f') = \widehat{S}_{\text{disc}}^{G'}(f')$, then $\widehat{S}^G(f') = \widehat{S}_{\text{disc}}^G(f')$ \square

We need to connect the distribution of the geometric side and the distribution of the spectral side in the stable local trace formula. We need the stable Weyl integral formula.

We know that

$$f_2^G(\phi) = \sum_{\pi \in \Pi_\phi} \Theta(\pi, f_2).$$

Substitute the Weyl integral formula [5] into this formula, we obtain an expansion,

$$f_2^G(\phi) = \sum_{\pi \in \Pi_\phi} \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Gamma_{\text{ell}}(M(\mathbb{R}), \zeta)} \Phi_M(\pi, \gamma) I_G(\gamma, f_2) d\gamma,$$

where $\Phi_M(\pi, \gamma) = |D(\gamma)|^{1/2} \Theta(\pi, \gamma)$, and $\Theta(\pi, \gamma) = \text{tr } \pi(\gamma)$.

We need to recall the basic object of geometric side before stabilizing the Weyl integral formula. We shall make free use of the language and notation [24] in this part, often without comment, to define the general transfer factor $\Delta(\sigma', \gamma)$, it is necessary to fix elements $\bar{\sigma}'$ and $\bar{\gamma}$ such that $\bar{\sigma}'$ is an image of $\bar{\gamma}$, and to specify $\Delta(\bar{\sigma}', \bar{\gamma})$. We will take it to be any complex number of absolute value 1, then $\Delta(\sigma', \gamma)$ is defined to be the product of $\Delta(\bar{\sigma}', \bar{\gamma})$ with the factor

$$\Delta(\sigma', \gamma; \bar{\sigma}', \bar{\gamma}) = \frac{\Delta_I(\sigma', \gamma)}{\Delta_I(\bar{\sigma}', \bar{\gamma})} \frac{\Delta_{II}(\sigma', \gamma)}{\Delta_{II}(\bar{\sigma}', \bar{\gamma})} \frac{\Delta_2(\sigma', \gamma)}{\Delta_2(\bar{\sigma}', \bar{\gamma})} \Delta_1(\sigma', \gamma; \bar{\sigma}', \bar{\gamma}).$$

There is an additional factor $\Delta_{IV}(\sigma', \gamma) = |D^G(\gamma)| |D^{G'}(\gamma')|^{-1}$ included in the definition of [24], but since we have already put these normalizing factors into our orbital integrals, the term does not appear in this equation. The remaining factors are all constructed from the special values of unitary abelian characters, and therefore have absolute value 1. There is a natural measure on $\Gamma_{\text{ell}}(G, \zeta)$ given by

$$\int_{\Gamma_{\text{ell}}(G, \zeta)} \alpha(\gamma) d\gamma = \sum_{\{T\}} |W(G(\mathbb{R}), T(\mathbb{R}))|^{-1} \int_{T(\mathbb{R})/Z(\mathbb{R})} \alpha(t) dt$$

for any $\alpha \in C(\Gamma_{\text{ell}}(G, \zeta))$, where $\{T\}$ is a set of representatives of $G(\mathbb{R})$ conjugacy classes of elliptic torus in G over \mathbb{R} , $W(G(\mathbb{R}), T(\mathbb{R}))$ is the Weyl group of $(G(\mathbb{R}), T(\mathbb{R}))$, and dt is the Haar measure on $T(\mathbb{R})$, $\Gamma_{\text{ell}}(G, \zeta)$ is the set of conjugacy classes γ in $G(\mathbb{R})$ such that G_γ is an elliptic maximal torus in G and as a distribution with a central character ζ on $Z(\mathbb{R})$ as in [9]. Let $\Delta_{\text{ell}}(G, \zeta)$ is the set of stable conjugacy classes in $\Gamma_{\text{ell}}(G, \zeta)$, we define a measure on $\Delta_{\text{ell}}(G, \zeta)$ by setting

$$\int_{\Delta_{\text{ell}}(G, \zeta)} \beta(\delta) d\delta = \sum_{\{T\}_{\text{stab}}} |W_{\mathbb{R}}(G, T)|^{-1} \int_{T(\mathbb{R})/Z(\mathbb{R})} \beta(t) dt$$

for any $\beta \in C(\Delta_{\text{ell}}(G, \zeta))$, where $\{T\}_{\text{stab}}$ is a set of representatives of stable conjugacy classes of elliptic maximal tori in G over \mathbb{R} . And $W_{\mathbb{R}}(G, T)$ is the subgroup of elements in the absolute Weyl group of (G, T) defined over \mathbb{R} . The measure on $\Gamma_{\text{ell}}(G, \zeta)$ and $\Delta_{\text{ell}}(G, \zeta)$ are related by a formula

$$(7.3) \quad \int_{\Delta_{\text{ell}}(G, \zeta)} \left(\sum_{\gamma \rightarrow \delta} \alpha(\gamma) \right) d\delta = \int_{\Gamma_{\text{ell}}(G)} \alpha(\gamma) d\gamma$$

Let $\Gamma_{\text{ell}}^{\mathcal{E}}(G, \zeta)$ be the set of isomorphism classes of pair (G', σ') , where G' is an elliptic endoscopic datum for G and σ' in an element in $\Delta_{G, \text{ell}}(G', \zeta)$. By an isomorphism from (G', σ') to the second pair (G'_1, σ'_1) , we mean an isomorphism from the datum G' to G'_1 , which takes σ' to σ'_1 . So we have a decomposition

$$\Gamma_{\text{ell}}^{\mathcal{E}}(G, \zeta) = \coprod_{G' \in \mathcal{E}_{\text{ell}}(G)} \Delta_{\text{ell}}(G', G, \zeta)$$

where $\Delta(G', G, \zeta) = \Delta_{G, \text{ell}}(G', \zeta) / \text{Out}_G(G')$. We also have an analogue of Lemma 6.2 for the geometric transfer factor.

Lemma 7.2. *Suppose that $\alpha \in C(\Gamma_{\text{ell}}(G, \zeta))$, and that $\beta \in C(\Gamma_{\text{ell}}^{\mathcal{E}}(G, \zeta))$, then*

$$\int_{\Gamma_{\text{ell}}(G, \zeta)} \sum_{\delta \in \Gamma_{\text{ell}}^{\mathcal{E}}(G, \zeta)} \beta(\delta) \Delta(\gamma, \delta) \alpha(\gamma) d\gamma = \int_{\Gamma_{\text{ell}}^{\mathcal{E}}(G, \zeta)} \sum_{\gamma \in \Gamma_{\text{ell}}(G, \zeta)} \beta(\delta) \Delta(\gamma, \delta) \alpha(\gamma) d\delta,$$

Proof. Let $\psi : G \rightarrow G^*$ be the underlying quasisplit inner twist of G . According to (7.3) the integral over $\Gamma_{\text{ell}}(G, \zeta)$ can be decomposed into an integral over $\delta^* \in \Delta_{\text{ell}}(G^*, \zeta)$ and a sum over the elements $\gamma \in \Gamma_{\text{ell}}(G, \zeta)$ which map to δ^* , similarly, the integral over $\Gamma_{\text{ell}}^{\mathcal{E}}(G, \zeta)$ can be decomposed into an integral over $\delta^* \in \Delta_{\text{ell}}(G^*, \zeta)$, and a summation over the element $\delta' \in \Gamma_{\text{ell}}^{\mathcal{E}}(G, \zeta)$ which map to δ^* , depends on the fact that the map $\delta' \rightarrow S_{T^*}(\delta')$, $T^* = G_{\delta^*}$ is a bijection from the preimage of δ^* in $\Gamma_{\text{ell}}^{\mathcal{E}}(G, \zeta)$ onto $\mathcal{K}(T^*)$, and this bijection depends on G is K -group.

With the two decomposition, we can represent each side of the required identity as an integral over $\Delta_{\text{ell}}(G^*, \zeta)$, and a double sum over δ and γ . The transfer factor $\Delta(\delta, \gamma)$ vanishes unless δ and γ have the same image in the $\Delta_{\text{ell}}(G^*, \zeta)$, the double sum in each case can therefore be taken over the preimages of δ^* in $\Gamma_{\text{ell}}^{\mathcal{E}}(G, \zeta) \times \Gamma_{\text{ell}}(G, \zeta)$, then identity follows. \square

With the following lemma, we stabilize the Weyl integral formula.

Lemma 7.3. *The Weyl integral formula is an expansion*

$$\Theta(\pi, f_2) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Gamma_{\text{ell}}(M(\mathbb{R}), \zeta)} \Phi_M(\pi, \gamma) I_G(\gamma, f_2) d\gamma,$$

then the stable Weyl integral formula is an expansion

$$f_2^G(\phi) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Gamma_{\text{ell}}^{\mathcal{E}}(M(\mathbb{R}), \zeta)} n(\delta)^{-1} \sum_{\pi \in \Pi_{\phi}} S\Phi_M(\pi, \delta) f_{2,M}^{\mathcal{E}}(\delta) d\delta$$

Proof. We observe that $I_G(\gamma, f_2) = I_M^M(\gamma, f_2) = f_{2,M}(\gamma)$ where $\gamma \in \Gamma_{\text{ell}}(M(\mathbb{R}), \zeta)$, and (7.4)

$$\Theta(\pi, f_2) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Gamma_{\text{ell}}(M(\mathbb{R}), \zeta)} n(\delta)^{-1} \Phi_M(\pi, \gamma) \sum_{\delta \in \Gamma_{\text{ell}}^{\mathcal{E}}(M(\mathbb{R}), \zeta)} \overline{\Delta(\delta, \gamma)} f_{2,M}^{\mathcal{E}}(\delta) d\gamma.$$

Apply 7.2, we have

$$\Theta(\pi, f_2) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Gamma_{\text{ell}}^{\mathcal{E}}(M(\mathbb{R}), \zeta)} n(\delta)^{-1} S\Phi_M(\pi, \delta) f_{2,M}^{\mathcal{E}}(\delta) d\delta,$$

where $S\Phi_M(\pi, \delta) = \sum_{\gamma \in \Gamma_{\text{ell}}(M(\mathbb{R}), \zeta)} \overline{\Delta(\delta, \gamma)} \Phi_M(\pi, \gamma)$. So we obtain

$$f_2^G(\phi) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Gamma_{\text{ell}}^{\mathcal{E}}(M(\mathbb{R}), \zeta)} n(\delta)^{-1} \sum_{\pi \in \Pi_{\phi}} S\Phi_M(\pi, \delta) f_{2,M}^{\mathcal{E}}(\delta) d\delta.$$

\square

We can give the stable distribution $S_M^G(\delta, f_1)$ by comparing with the stable Weyl integral formula.

Theorem 7.4. *Let $f_1 \in C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, then*

$$(7.5) \quad S_M^G(\delta, f_1) = (-1)^{\dim(A_M/A_G)} \int_{\Phi_2(G, \zeta)} S^G(\phi) S\Phi_M(\phi, \delta) \tilde{f}_1^G(\phi) d\phi,$$

where

$$S\Phi_M(\phi, \delta) = \sum_{\pi \in \Pi_\phi} \sum_{\gamma \in \Gamma_{\text{ell}}(M, \zeta)} \overline{\Phi_M(\pi, \gamma)} \Delta(\delta, \gamma),$$

and

$$\Phi_M(\pi, \gamma) = \begin{cases} |D(\gamma)|^{1/2} \Theta(\pi, \gamma) & \text{if } \gamma \in M(\mathbb{R})_{\text{ell}}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{f}_1^G(\phi) = \sum_{\tau \in T_{\text{ell}}(G, \zeta)} \Delta(\phi, \tau) |d(\tau)|^{-1} \Theta(\tau, f_1),$$

for any group $M \in \mathcal{L}$ and δ is stable strongly G -regular class in $M(\mathbb{R})$

Proof. Suppose that δ does not lie in $\Delta_{\text{ell}}(M(\mathbb{R}), \zeta)$, by descent formula [8, §6] and the cuspidality of f_1 , then $S_M^G(\delta, f_1)$ vanishes. The right hand side of (7.5) vanishes by definition. So the formula holds in the case. It is therefore enough to establish (7.5) where δ lies in $\Delta_{\text{ell}}(M(\mathbb{R}), \zeta)$.

To deal with elliptic point in $M(\mathbb{R})$, we apply the simple version of the stable local trace formula. Consider the two expression (7.1) and (7.2) in the 7.1, with f_1 the given cuspidal function and f_2 a variable function in $C(G(\mathbb{R}), \zeta)$. The expressions depend on f_2 through different distributions f_2^G and $f_{2,M}^\mathcal{E}$. However, the relation is given by a stable Weyl integral formula, which has an expansion

$$f_2^G(\phi) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Gamma_{\text{ell}}^\mathcal{E}(M(\mathbb{R}), \zeta)} n(\delta)^{-1} \sum_{\pi \in \Pi_\phi} S\Phi_M(\pi, \delta) f_{2,M}^\mathcal{E}(\delta) d\delta.$$

We obtain

$$f_2^G(\phi) = \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \int_{\Gamma_{\text{ell}}^\mathcal{E}(M(\mathbb{R}), \zeta)} n(\delta)^{-1} S\Phi_M(\phi, \delta) f_{2,M}^\mathcal{E}(\delta) d\delta.$$

Substituting this into (7.2), we collect the coefficient of $\overline{f_2^M(\delta)}$ in the resulting identity of (7.1) with (7.2). We see that if

$$P_M(\delta, f_1) = S_M^G(\delta, f_1) - (-1)^{\dim(A_M/A_G)} \int_{\Phi_2(G, \zeta)} S^G(\phi) S\Phi_M(\phi, \delta) \tilde{f}_1^G(\phi) d\phi,$$

then the sum of

$$(7.6) \quad \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Delta_{G, \text{ell}}(M, \zeta)} n(\delta)^{-1} P_M(\delta, f_1) \overline{f_2^M(\delta)} d\delta,$$

and

(7.7)

$$- \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} \sum_{M' \in \mathcal{E}_{\text{ell}}^0(M)} \int_{\Delta_{\text{ell}}(M', M, \zeta)} \int_{\Phi_2(G, \zeta)} n(\delta)^{-1} S^G(\phi) \widetilde{f}_1^G(\phi) S\Phi_M(\phi, \delta) \overline{f_2^{M'}(\delta)} d\delta d\phi$$

vanishes.

We have written $f_{2,M}^\mathcal{E}(\delta') = f_2^{M'}(\delta')$, where $\delta' \in \Delta_{G,\text{ell}}(M', \zeta)$. As in [8, §10], we choose f_2 so that $f_{2,G}^\mathcal{E}$ has compact support modulo $Z(\mathbb{R})$ on $\Gamma^\mathcal{E}(G) = \coprod_{\{M\}} \Gamma_{\text{ell}}^\mathcal{E}(M(\mathbb{R}), \zeta)$, and so that $f_{2,G}^\mathcal{E}$ approaches the ζ^{-1} -equivariant Dirac measure at the image of δ' in $\Gamma^\mathcal{E}(G)$. The expression then approaches a nonzero multiple of $P_M(\delta', f_1)$ when $\delta' \in \Delta_{G,\text{ell}}(M, \zeta)$, and approaches a zero. Then vanishes, and we conclude that $P_M(\delta, f_1) = 0$. So we have obtained the required formula

$$S_M^G(\delta, f_1) = (-1)^{\dim(A_M/A_G)} \int_{\Phi_2(G, \zeta)} S^G(\phi) S\Phi_M(\phi, \delta) \widetilde{f}_1^G(\phi) d\phi.$$

□

Corollary 7.5. *If we set $S_M^G(M', \delta, f) = n(\delta)^{-1} \int_{\Phi_2(G, \zeta)} S^G(\phi) S\Phi_M(\phi, \delta) \widetilde{f}^G(\phi) d\phi$, and $\delta \in \Delta_{\text{ell}}(M', M, \zeta)$. Then $S_M^G(M', \delta, f)$ vanishes.*

The proof of the corollary comes from the process of proof 7.4. We have obtained the main terms about the multiplicity, we need to know the relation of stable distribution $S_M^G(\delta, f_1)$ and the invariant distribution $\Phi_M(\gamma, f_1)$.

8. MULTIPLICITY FORMULA OF DISCRETE SERIES

We now return to the discussion of section 3. In order to establish a multiplicity formula of discrete series, we descend from the K -group to the connected reductive group, if we take the test function whose components vanish except for the one from the required connected group, then we can apply the properties of K -group to connected reductive group. We assume the infinitesimal character μ is regular. We take the center $Z(\mathbb{R}) = Z(G(\mathbb{R}))$, then $a_{G,Z}^* = 1$, and the stable distribution $S_M^G(\delta, f_1)$ is simple. If μ is regular, then π_μ is discrete series, the L -packet $\Pi_\phi = \mathcal{E}(T_{M_\phi})$, and R_ϕ is trivial. So $|i^G(\tau)|$ is trivial, and $\tau = \pi \in \Pi_2(G(\mathbb{R}), \zeta)$, then

$$\begin{aligned} \widetilde{f}_1^{G'}(\phi') &= \sum_{\tau \in T_{\text{ell}}(G, \zeta)} \Delta(\phi', \tau) |i^G(\tau)| \Theta(\tau, f_1) \\ (8.1) \quad &= \sum_{\pi \in \Pi_2(G(\mathbb{R}), \zeta)} \Delta(\phi', \pi) \text{tr } \pi(f_1) \\ &= f_1^{G'}(\phi') \end{aligned}$$

and $f_1^{G'}(\phi') = \sum_{\pi \in \Pi_{\phi'}} \Theta(\pi, f_1)$.

Now our main obstruction is that the pseudo-coefficient f_{π_μ} is cuspidal, but not stable. However f_{π_μ} transfers to $f'_{\phi'_\mu}(\phi')$ on G' by Shelstad transfer theorem, then $f'_{\phi'_\mu}(\phi')$ is stable cuspidal, where $f'_{\phi'_\mu}(\phi') = \sum_{\pi \in \Pi_2(G(\mathbb{R}), \zeta)} \Delta(\phi', \pi) \text{tr} \pi(f_{\pi_\mu}) = \Delta(\phi', \pi_\mu)$.

Then

$$f'_{\phi'_\mu}(\phi') = \begin{cases} \Delta(\phi'_\mu, \pi_\mu) & \text{if } \phi' = \phi'_\mu, \\ 0 & \text{otherwise.} \end{cases}$$

We take $f'_{\phi'_\mu}(\phi') = \Delta(\phi', \pi_\mu) f_{\phi'_\mu}$ and $f_{\phi'_\mu} = \sum_{\pi \in \Pi_{\Phi'_\mu}} f_\pi$, where f_π is pseudo-coefficient. Then Arthur's result in [3] can work on the endoscopic group G' of G . We assume that the test function f is stable cuspidal on G' , and denote G' for G , which is a quasisplit group.

We get a simple stable distribution from (7.5),

$$S_M^G(\delta, f) = (-1)^{\dim(A_M/A_G)} \sum_{\phi \in \Phi_2(G, \zeta)} S^G(\phi) S\Phi_M(\phi, \delta) f^G(\phi),$$

where

$$S\Phi_M(\phi, \delta) = \sum_{\pi \in \Pi_\phi} \sum_{\gamma \in \Gamma_{\text{ell}}(M, \zeta)} \Delta(\delta, \gamma) \overline{\Phi_M(\pi, \gamma)},$$

$$\Phi_M(\pi, \gamma) = |D^G(\gamma)|^{1/2} \Theta_\pi(\gamma) = I_M(\pi, \gamma)$$

in [3], and

$$f^G(\phi) = \sum_{\pi \in \Pi_\phi} \text{tr} \pi(f) = |\mathcal{S}_\phi| \overline{\text{tr} \tilde{\pi}(f)}.$$

So

$$S_M^G(\delta, f) = (-1)^{\dim(A_M/A_G)} \sum_{\phi \in \Phi_2(G, \zeta)} \sum_{\pi \in \Pi_\phi} \sum_{\gamma \in \Gamma_{\text{ell}}(M, \zeta)} |\mathcal{S}_\phi| S^G(\phi) \Delta(\delta, \gamma) \overline{I_M(\pi, \gamma) \text{tr} \tilde{\pi}(f)}.$$

where

$$|\mathcal{S}_\phi| S^G(\phi) = |\mathcal{S}_\phi| |\mathcal{S}_\phi|^{-1} = 1.$$

However,

$$\begin{aligned} \sum_{\phi \in \Phi_2(G, \zeta)} \sum_{\pi \in \Pi_\phi} \overline{I_M(\pi, \gamma) \text{tr} \tilde{\pi}(f)} &= \sum_{\pi \in \Pi_2(G(\mathbb{R}), \zeta)} \overline{I_M(\pi, \gamma) \text{tr} \tilde{\pi}(f)} \\ &= (-1)^{\dim(A_M/A_G)} \text{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0) \overline{I_M(\gamma, \bar{f})} \\ &= (-1)^{\dim(A_M/A_G)} |D^M(\gamma)|^{1/2} \text{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0) \overline{\Phi_M(\gamma, \bar{f})}. \end{aligned}$$

Then we obtain the following theorem.

Theorem 8.1. *Suppose that $f \in C(G(\mathbb{R}), \zeta)$ is stable cuspidal and that $\delta \in M(\mathbb{R})$, then*

$$\begin{aligned} S_M^G(\delta, f) &= (-1)^{\dim(A_M/A_G)} \sum_{\phi \in \Phi_2(G, \zeta)} S^G(\phi) S\Phi_M(\phi, \delta) f^G(\phi) \\ &= \sum_{\gamma \in \Gamma_{\text{ell}}(M, \zeta)} \Delta(\delta, \gamma) |D^M(\gamma)|^{1/2} \text{vol}(T(\mathbb{R})/A_M(\mathbb{R})^0) \overline{\Phi_M(\gamma, \bar{f})}. \end{aligned}$$

In particular, $S_M^G(\delta, f)$ vanishes, if δ is not semisimple.

Proof. we just need to check that the stable distribution $S_M^G(\delta, f)$ vanishes, when δ is not semi-simple. We know $\Phi_M(\gamma, \bar{f})$ vanishes, if γ is not semisimple, and $f \in \mathcal{H}_{\text{ac}}(G(\mathbb{R}), \zeta)$ (Arthur studied this space in [3]) is stable cuspidal. However, because $\mathcal{H}_{\text{ac}}(G(\mathbb{R}), \zeta)$ is dense in $C(G(\mathbb{R}), \zeta)$, we can use the trace Paley-Wiener theorem to extend the result to $C(G(\mathbb{R}), \zeta)$. The other part comes from Proposition 4.2. \square

We denote $S\Phi_M^G(\delta, f) = |D^M(\delta)|^{-\frac{1}{2}} S_M^G(\delta, f)$. From the above theorem, we have defined $S\Phi_M^G(\delta, f)$ on the strongly regular points of $M(\mathbb{R})$ by the stable conjugacy class, which depends on γ . $\Delta(\delta, \gamma)$ is a continuous function on δ and the point $\gamma \in M(\mathbb{R})$. $\Phi(\gamma, f)$ is a continuous function on the maximal torus $T(\mathbb{R})$, so we can extend $S\Phi_M^G(\delta, f)$ to a continuous function on $T(\mathbb{R})$. If δ stable conjugacy of $\delta_1 \in T(\mathbb{R})$, then $S\Phi_M^G(\delta, f) = S\Phi_M^G(\delta_1, f)$, otherwise $S\Phi_M^G(\delta, f)$ vanishes. So $S\Phi_M^G(\delta, f)$ is a continuous function on M .

We can give a dimension formula for spaces of automorphic forms. For each $\pi_{\mathbb{R}} \in \Pi_2(G(\mathbb{R}), \zeta)$, let $m_{\text{disc}}(\pi_{\mathbb{R}}, K_0)$ be the multiplicity of $\pi_{\mathbb{R}}$ in

$$(8.2) \quad L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_0, \zeta) = \bigoplus_{i=1}^n L^2(\Gamma_i \backslash G(\mathbb{R}), \zeta),$$

where K_0 is an open compact subgroup of $G(\mathbb{A}_{\text{fin}})$, and $\{\Gamma_i\}$ are the discrete subgroup. Let h be a K_0 bi-invariant function in $\mathcal{H}(G(\mathbb{A}_{\text{fin}}))$. Let $R_{\text{disc}}(\pi_{\mathbb{R}}, h)$ be the operator on $\pi_{\mathbb{R}}$ -isotypical subspace of (8.2), it can be interpreted as a $\left(m_{\text{disc}}(\pi_{\mathbb{R}}, K_0) \times m_{\text{disc}}(\pi_{\mathbb{R}}, K_0) \right)$ matrix.

Then 4.1 yields the formula,

$$\begin{aligned} & \text{tr}(R_{\text{disc}}(\pi_{\mathbb{R}}, h)) = I(f_{\pi_{\mu}} h) \\ &= \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \sum_{M' \in \mathcal{L}^{G'}} |W_0^{M'}| |W_0^{G'}|^{-1} \sum_{\delta \in \{M'(\mathbb{Q})\}_{M', S}} b^{M'}(\delta) S_{M'}^{G'}(\delta, f_{\pi_{\mu}}) (h_M)^{M'}(\delta), \end{aligned}$$

where $\pi_{\mathbb{R}} = \pi_{\mu}$, $M \in \mathcal{L}(G)$ and $M' \in \mathcal{E}(M)$. The function $f_{\pi_{\mu}}$ is a pseudo-coefficient, which belongs to $C_{\text{cusp}}(G(\mathbb{R}), \zeta)$, and

$$S_{M'}^{G'}(\delta, f_{\pi_{\mu}}) = \widehat{S}_{M'}^{G'}(\delta, f'_{\phi'_{\mu}}) = (-1)^{\dim(A_{M'}/A_{G'})} S^{G'}(\phi'_{\mu}) S\Phi_{M'}(\phi'_{\mu}, \delta) \Delta(\phi'_{\mu}, \pi_{\mu}).$$

In particular, the function vanishes unless δ is semisimple, and the set of equivalence classes in $\{M(\mathbb{Q})\}_{M, S}$ which are just $M(\mathbb{Q})$ -semisimple stable conjugacy classes. Moreover, for any semisimple elliptic stable conjugacy class $\delta \in \{M(\mathbb{Q})\}_{\text{ell}}$, we have the global coefficient $b^{M'}(\delta) = b_{\text{ell}}^{M'}(\delta) = \tau(M')$.

We discuss the transfer of $h_M(\gamma)$. For the given function $h \in \mathcal{H}(G(\mathbb{A}_{\text{fin}}))$, we can find a function $h^{G'} \in \mathcal{H}(G'(\mathbb{A}_{\text{fin}}))$ whose orbital integrals match those of h . In other words we need $h^{G'} \in \mathcal{H}(G'(\mathbb{A}_{\text{fin}}))$ such that for all $\delta \in G'_{\text{reg}}(\mathbb{A}_{\text{fin}})$

$$(h_M)^{M'}(\delta) = \sum_{\gamma} \Delta(\delta, \gamma) h_M(\gamma),$$

where the sum is taken over $M(\mathbb{A}_{\text{fin}})$ -conjugacy classes of images $\gamma \in M(\mathbb{A})_{\text{fin}}$ of δ . It follows from Ngô's proof of the fundamental lemma, and Waldspurger [32], [33] contribution

of the implication "the fundamental lemma implies the transfer", that the function $(h_M)^{M'}$ always exists. Here

$$h_M(\gamma) = |D^M(\gamma)|_{\mathbb{R}}^{-1/2} \delta_P(\gamma_{\text{fin}})^{1/2} \int_{K_{\text{fin}}} \int_{N_P(\mathbb{A}_{\text{fin}})} \int_{M_\gamma(\mathbb{A}_{\text{fin}}) \backslash M(\mathbb{A}_{\text{fin}})} h(k^{-1}m^{-1}\gamma mnk) dmdndk.$$

Here $P = MN_P$ is any parabolic subgroup with Levi component M , $\delta_P(\gamma_{\text{fin}})$ is the modular function of P , evaluated at the image of γ in $G(\mathbb{A}_{\text{fin}})$. In particular, the stable orbit integral on M' can be taken over $M'(\mathbb{A}_{\text{fin}})$ rather than $M'(\mathbb{Q}_{S_0})$, here $S_0 = S - \{\infty\}$, we thus have no further need to single out the finite set of valuations.

Remark 8.2. I thank Waldspurger for pointing out the following fact. The fundamental lemma is proved in the unramified case and if the residual characteristic p is big. If the situation is not unramified, the problem is more complicated. There are several conjugacy classes of maximal compact subgroups, and there is no natural correspondence between maximal compact subgroups of G and maximal subgroups of G' . So we cannot expect a simple formula for the transfer of the characteristic function of some maximal compact subgroup of G . Maybe, we can hope that this transfer is a linear combination of characteristic functions of maximal compact subgroups of G' .

We set

$$P_\mu(M') = S^{G'}(\phi'_\mu) \Delta(\phi'_\mu, \pi_\mu) T(M')$$

and we also write $\{M(\mathbb{Q})\}$ for the set of $M(\mathbb{Q})$ -semisimple stable conjugacy classes in $M(\mathbb{Q})$. We have obtained the main theorem.

Theorem 8.3. *If $h \in \mathcal{H}(G(\mathbb{A}_{\text{fin}}))$, and the highest weight μ of representation is regular, then we have*

$$\begin{aligned} & \text{tr}(R_{\text{disc}}(\pi_{\mathbb{R}}, h)) \\ &= \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \sum_{M' \in \mathcal{L}^{G'}} (-1)^{\dim(A_{M'}/A_{G'})} |W_0^{M'}| |W_0^{G'}|^{-1} \sum_{\delta \in \{M'(\mathbb{Q})\}} P_\mu(M') S\Phi_{M'}(\phi'_\mu, \delta) (h_M)^{M'}(\delta), \end{aligned}$$

and the multiplicity formula of the discrete series

$$m_{\text{disc}}(\pi_{\mathbb{R}}, K_0) = \text{tr}(R_{\text{disc}}(\pi_{\mathbb{R}}, 1_{K_0})).$$

Remark 8.4. (1) The sum in δ can be taken over a finite sum that depends only on the support of h , so the theorem therefore provides a finite closed formula for $\text{tr}(R_{\text{disc}}(\pi_{\mathbb{R}}, h))$.

(2) Naturally, one could ask whether the stable trace formula have a geometric explanation. What is the geometric object corresponding to the stable coefficient $P_\mu(M)$? I think this is a very interesting problem, because we want to find a way to obtain a more general trace formula which can attack the beyond endoscopy. And in the function field case, Frenkel-Ngo have proposed that the trace formula is a consequence of Atiyah-Bott Lefschetz fixed point formula, applied to Hitchin moduli space.

9. THE STABLE FORMULA OF L^2 -LEFSCHETZ NUMBER

We can give a stable formula for L^2 -Lefschetz number. We can refer to the Arthur's paper [3], which gives a formula through the invariant trace formula. This part, we will use our result to obtain the stable formula. We need to recall the basic result in [3]. We still use the notation in [3]. If $h \in \mathcal{H}(G(A_{\text{fin}}))$, then

$$\begin{aligned}
 \mathcal{L}_\mu(h) &= \sum_q (-1)^q \text{tr}(H_2^q(h, \mathcal{F}_\mu)) \\
 &= \sum_{\pi \in \Pi(G(\mathbb{A}, \xi))} m_{\text{disc}}(\pi) \chi_\mu(\pi_{\mathbb{R}}) \text{tr } \pi_{\text{fin}}(h) \\
 (9.1) \quad &= \sum_{\pi \in \Pi(G(\mathbb{A}, \xi))} m_{\text{disc}}(\pi) \text{tr } \pi_{\mathbb{R}}(f_\mu) \text{tr } \pi_{\text{fin}}(h) \\
 &= \sum_{\pi \in \Pi(G(\mathbb{A}), \xi)} m_{\text{disc}}(\pi) \text{tr}(f_\mu h) \\
 &= I(f_\mu h)
 \end{aligned}$$

where $\chi_\mu(\pi_{\mathbb{R}}) = \sum_q (-1)^q \dim H^q(g(\mathbb{R}), K'_R; \pi_{\mathbb{R}} \otimes \mu)$, and

$$(9.2) \quad \chi_\mu(\pi_{\mathbb{R}}) = \text{tr } \pi_{\mathbb{R}}(f_\mu) = \begin{cases} (-1)^{q(G)} & \text{if } \pi_{\mathbb{R}} \in \Pi_{\text{disc}}(\tilde{\mu}), \\ 0 & \text{otherwise.} \end{cases}$$

If the test function is stable cuspidal, then the unipotent distribution vanishes, and only semi-simple distribution contributes in the trace formula. We can obtain the explicit invariant distribution and invariant trace formula. So we have

$$I(f_\mu h) = \sum_{M \in \mathcal{L}} (-1)^{\dim(A_M/A_G)} |W_0^M| |W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}))} \chi(M_\gamma) |\iota^M(\gamma)|^{-1} \Phi_M(\gamma, \mu) h_M(\gamma)$$

where $\chi(M_\gamma) = (-1)^{q(M_\gamma)} \text{vol}(M_\gamma(\mathbb{Q}) A_{M_\gamma}(\mathbb{R})^\circ \backslash M_\gamma(\mathbb{A})) \text{vol}(A_{M_\gamma}(\mathbb{R})^\circ \backslash \overline{M_\gamma(\mathbb{A})^+}) |\mathcal{D}(M_\gamma, B)|$, $\mathcal{D}(G, B) = W(G(\mathbb{R}), B(\mathbb{R})) \backslash W(G, B)$, $|\iota^M(\gamma)| = |M_\gamma(\mathbb{Q}) \backslash M(\mathbb{Q}, \gamma)|$, and $(M(\mathbb{Q}))$ are the $M(\mathbb{Q})$ -conjugacy classes of $M(\mathbb{Q})$.

Since f_μ is stable cuspidal, we can use the Proposition 4.1. Then we obtain the stable formula

$$I(f_\mu h) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \sum_{M' \in \mathcal{L}^{G'}} |W_0^{M'}| |W_0^{G'}|^{-1} \sum_{\delta \in \Delta(M', V, \zeta)} b^{M'}(\delta) S_{M'}^{G'}(\delta, f_\mu) (h_M)^{M'}(\delta).$$

Where $S_{M'}^{G'}(\delta, f_\mu) = (-1)^{\dim(A_{M'}/A_{G'})} \sum_{\phi' \in \Phi_2(G', \xi)} S^{G'}(\phi') S \Phi_M(\phi', \delta) f_\mu^{G'}(\phi')$ and

$$(9.3) \quad f_\mu^{G'}(\phi') = \sum_{\pi \in \Pi_2(G(R), \xi)} \Delta(\phi', \pi) \text{tr } \pi(f_\mu) = \begin{cases} (-1)^{q(G)} \sum_{\pi \in \Pi_\phi} \Delta(\phi', \pi) & \text{if } \phi = \phi(\mu, \lambda), \\ 0 & \text{otherwise.} \end{cases}$$

where $q(G) = \frac{1}{2} \dim(G(\mathbb{R})/K_{\mathbb{R}})$, and $f_{\mu}^{G'}$ is a stable cuspidal function. We obtain the stable distribution

$$S_{M'}^{G'}(\delta, f_{\mu}) = (-1)^{\dim(A_{M'}/A_{G'})+q(G)} \sum_{\pi \in \Pi_{\phi(\mu, \lambda)}} \Delta(\phi', \pi) S^{G'}(\phi') S\Phi_{M'}(\phi', \delta).$$

Remark 9.1. If $G = G' = G^{s=1}$ is a quasisplit group, and $\Delta(\phi^s, \pi) = \xi(s)$ is just a character [29], we have the simple formula for the stable distribution.

$$(9.4) \quad f_{\mu}^{G^s}(\phi') = \begin{cases} (-1)^{q(G)} |\mathcal{S}_{\phi}| & \text{if } s = 1 \text{ and } \phi = \phi(\mu, \lambda), \\ 0 & \text{otherwise.} \end{cases}$$

where $|\mathcal{S}_{\phi}|$ is the cardinal number of the L -packet of ϕ .

Since $f_{\mu}^{G'}$ is stable cuspidal, we have $S_{M'}^{G'}(\delta, f_{\mu}) = \widehat{S}_{M'}^{G'}(\delta, f_{\mu}^{G'}) = 0$, if δ is a not semisimple element of M' . So we only compute the semisimple element coefficient. $b^{M'}(\delta) = \tau(M')$, if δ is a semisimple, elliptic element. However we also directly obtain this coefficient using the remark. When G is quasisplit, we assume that the test function is pseudo-coefficient, then the invariant trace formula and stable trace formula have the nature relations. Combine the Langlands-Shelstad transfer theorem, we can obtain the coefficient $b^M(\delta)$.

The set of equivalence classes in $\Delta(M', V, \xi)$ equals the set of equivalence classes in $\Delta(\bar{M}', V)$, which are just \bar{M}' -strongly regular stable conjugacy classes. We set

$$F_{\mu}(M') = (-1)^{\dim(A_{M'}/A_{G'})+q(G')} \tau(M') S^G(\phi') \sum_{\pi \in \Pi_{\phi(\mu, \lambda)}} \Delta(\phi', \pi),$$

where $(-1)^{q(G')} = (-1)^{q(G)}$, and we also write $\{M(\mathbb{Q})\}$ for the set of stable $M(\mathbb{Q})$ -conjugacy classes in $M(\mathbb{Q})$. The other main result is the following theorem.

Theorem 9.2. For any $h \in \mathcal{H}(G(\mathbb{A}_{\text{fin}}))$, we have

$$\mathcal{L}_{\mu}(h) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \sum_{M' \in \mathcal{L}^{G'}} |W_0^{M'}| |W_0^{G'}|^{-1} \sum_{\delta \in \{M'(\mathbb{Q})\}} F_{\mu}(M') S\Phi_{M'}(\phi', \delta) (h_M)^{M'}(\delta).$$

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MORNINGSIDE CENTER OF MATHEMATICS ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE,
CHINESE ACADEMY OF SCIENCES NO. 55 ZHONGGUANCUN DONGLU, HAIDIAN DISTRICT BEIJING
100190 P.R. CHINA.

E-mail address: pengzhifeng1982@163.com